Planetary and Space Science 59 (2011) 355–363

Contents lists available at ScienceDirect

Planetary and Space Science

journal homepage: www.elsevier.com/locate/pss

Model approximation of cosmic ray spectrum

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ARTICLE INFO

Article history:
Received 25 July 2010
Received in revised form
26 December 2010
Accepted 3 January 2011
Available online 11 January 2011

Keywords:
Cosmic rays
Modeling
Heliospheric modulation

ABSTRACT

An analytical model which generalizes the equations describing the intensity of galactic cosmic rays (CR), including both processes, making it applicable in the inner heliosphere (where energy losses dominate) and outer heliosphere (influenced primarily by convection–diffusion processes) is derived. By a suitable choice of a parameter, the proposed model turns into two approximations: solution close to "force–field" model (describing the energy losses of CR in the inner heliosphere) and "convection–diffusion" equation (giving the reduction of CR intensity in the outer heliosphere). A mathematical relation between parameters in the proposed model and the modulation parameter Φ is derived.

1. Introduction

The galactic cosmic rays (CR) are an important factor for the physics of the Earth environment and the interplanetary space. They have significant influence on the space weather (Kudela et al., 2000; Dorman, 2004; Belov et al., 2004; Storini, 2006) and also on the atmosphere and ionosphere (Dorman, 2004; Storini, 2006; Eroshenko et al., 2009). That is why the modeling of the differential spectrum of galactic CR is of fundamental importance in modeling the variety of physical and chemical processes in the atmosphere.

The transport of galactic cosmic rays is often described in terms of the spherically symmetric transport equation for the differential number density $U(r, E)$ with the following simplifying assumptions: a steady state flux and no sources of cosmic rays (Urch and Gleeson, 1972; cf. Eq. (2.4))

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2VU-r^2k\frac{\partial U}{\partial r}\right)-\frac{1}{3r^2}\frac{\partial}{\partial r}(r^2V)\frac{\partial}{\partial E}(\alpha_{\mathrm{tf}}EU)=0 \tag{1}$$

where r is heliocentric distance and E is the particle kinetic energy. The solar wind speed is given by $V(r)$, the particle diffusion coefficient by $k(r, E)$, and $\alpha_{\mathrm{tf}}(E)=(E+2E_0)/(E+E_0)$, with E_0 the rest energy of a particle (α_{tf}—troublesome factor (Moraal and Potgieter, 1982)). The terms in Eq. (1) describe, from left to right, the convection, diffusion of the particles and energy loss in the expanding solar wind. Eq. (1) can also be written in a form (Gleeson, 1971; cf. Eqs. (1) and (2))

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2S)=-\frac{V}{3}\frac{\partial^2}{\partial r\partial E}(\alpha_{\mathrm{tf}}EU)=0 \tag{2}$$

$$S=CVU-k\frac{\partial U}{\partial r} \tag{3}$$

where

$$C=1-\frac{1}{3U}\frac{\partial}{\partial E}(\alpha_{\mathrm{tf}}EU)$$

is Compton–Getting coefficient. When the energy loss term in Eq. (1) is expressed in terms of C, the equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2k\frac{\partial U}{\partial r}\right)-\frac{C}{r^2}\frac{\partial}{\partial r}(r^2V)U-V\frac{\partial U}{\partial r}=0 \tag{4}$$

2. Approximations to cosmic ray transport equation

Cosmic ray modulation is studied with various approximations of Eq. (1), which lead to solutions. We examine the most commonly used analytical approximations: "force–field (FF)" and "convection–diffusion" solutions for galactic cosmic ray particles. The force–field approximation describes the modulation as energy losses, while the convection–diffusion approximation describes it as a reduction in intensity.

The convection–diffusion approximation is obtained by neglecting the third term in Eq. (1) and it can be written as

$$VU=k\frac{\partial U}{\partial r} \tag{5}$$

If we assume that V/k is zero beyond a modulation boundary at heliocentric distance R, Eq. (5) has the following solution:

$$U(r, E)=U_{LIS}(R, P_R)\exp(-M), \tag{6}$$

where U_{LIS} is the differential number density of galactic cosmic rays at distance $r=R$ and P_R is the particle rigidity at distance R.

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doi:10.1016/j.pss.2011.01.001

The quality M is defined as (Gleeson and Axford, 1968; Caballero-Lopez and Moraal, 2004; cf. Eq. (17))

$$M=\int_r^R \frac{V}{k(r',P)}\,dr' \tag{7}$$

The integral M is the modulation function, and it is dimensionless quantity. For completeness, we mention that the cosmic ray intensity, $D(E)$, is related to the differential number density $U(E)$ by relation $D(E)=vU(E)/4\pi$, where v is the particle speed.

Compton–Getting coefficient in energy (or rigidity) in terms of the omnidirectional distribution function $f(P)$ as a function of particle rigidity is given by (Forman, 1970; Fisk et al., 1973; cf. Eq. (4))

$$C=-\frac{P}{3f}\frac{\partial f}{\partial P} \tag{8}$$

The function f represents the number of particles per unit volume of the phase space averaged over the particle direction. The distribution function $f(P)$ is related to the cosmic ray intensity $D(E)$ by

$$D(E)=c(A/Z)P^2 f(P), \tag{9}$$

where c is the speed of light, and A and Z are mass and charge numbers, respectively.

The energy variable in Eq. (1) may be changed to rigidity, P with $P=pc/Z=A/Z(E(E+2E_0))^{1/2}$. Then, Eq. (1) or Eq. (4) in terms of $f(P)$ becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k\frac{\partial f}{\partial r}\right)-V\frac{\partial f}{\partial r}+\frac{P}{3}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2V)\frac{\partial f}{\partial P}=0 \tag{10}$$

The streaming S can be neglected in Eq. (3) at relatively high energies, and a simple first order equation in terms of $f(P)$ known as the *force–field* (*FF*) *equation* is derived (Gleeson and Axford, 1968; Fisk et al., 1973; cf. Eq. (6))

$$\frac{\partial f}{\partial r}+\frac{VP}{3k}\frac{\partial f}{\partial P}=0 \tag{11}$$

The solution to Eq. (11) is $f(r,P)=f_0(R,P_R(r,P))$ along contours of the characteristic equation

$$\frac{dP}{dr}=\frac{VP}{3k} \tag{12}$$

in (r, P) space; $f_0(R, P_R)$ is the unmodulated distribution function determined at radius $r=R$, where the modulation is negligibly small. The rigidity $P_R=P_R(r;\ P)$ is obtained by integrating the characteristic equation from the initial phase space point $(r;\ P)$ to the point $(R;\ P_R)$ at the outer boundary R. If the diffusion coefficient is separable in the form $k=\overline{v}k_1(r)k_2(P)$, Eq. (12) becomes

$$\int_P^{P_R}\frac{\overline{v}(P')k_2(P')}{P'}\,dP'=\int_r^R\frac{V(r')}{3k_1(r')}\,dr'\equiv\varphi(r) \tag{13}$$

where φ (r) is called the force–field parameter (or modulation potential). $\overline{v}=v/c$=(Particle speed/speed of light). If k_2 $(P)\sim P$ and $\overline{v}\approx 1$, the solution reduces to a simple form

$$P_R-P=\varphi \tag{14}$$

In this case, the modulation potential φ (in GV) becomes a rigidity (or an energy) loss. Note that an FF approximation can be applied in the form (Eq. (14)) only to the special case of relativistic particles ($\overline{v}=1$) and rigidity dependence $k\sim P$.

Gleeson and Urch (1973) point out that the full FF parameter is $\varphi/k_2(P)$. From Eq. (13), follows that this ratio is expressed by

$$\frac{\phi}{\overline{v}k_2}=\int_r^R\frac{V}{3k}\,dr' \tag{15}$$

For relativistic particles Eq. (15) is written in the form

$$\frac{\phi}{k_2}=\int_r^R\frac{V}{3k}\,dr' \tag{15a}$$

Fisk and Axford (1969) and Fisk et al. (1973) showed that the force–field equation must be satisfied for small $\tilde{V}\tilde{r}/\tilde{k}\ll 1$ and average modulation level $\tilde{V}\tilde{r}/\tilde{k}<1$, where $\tilde{V}$, $\tilde{r}$ and $\tilde{k}$ are characteristic values of the solar wind speed, the heliocentric distance, and the diffusion coefficient, respectively.

3. Trigonometric parameterization of galactic cosmic ray spectrum *D*(*E*)

The differential intensity spectrum $D(E)$ in Fig. 1 can be presented by the trigonometric dependence

$$\beta=\frac{\ln(D_{LIS}(E))-\ln(D(E))}{\ln(E+\alpha)-\ln(E)} \tag{16}$$

where $\beta=\tan\tau$, and $D_{\rm LIS}(E)$ is the local interstellar spectrum. After some transformations including antilog ones Eq. (16) can be written in the form

$$D(E)=D_{LIS}(E)\left(1+\frac{\alpha}{E}\right)^{-\beta} \tag{17}$$

where α=const and $\beta=\tan\tau$=const. Here, $D_{\rm LIS}(E)$ is defined by

$$D_{LIS}(E)=K(E+E_0)^{-\gamma} \tag{17a}$$

where γ is the differential spectral index of the cosmic ray flux and K is the normalization constant. We consider the expression (Eq. (17)) for the following limits: (1) at energies $E\ll E_0$ $D_{\rm LIS}(E)$ is approximately constant and the differential spectrum $D(E)$ as a function of the energy E is described primarily by the second multiplier in Eq. (17) as $(1+\frac{\alpha}{E})^{-\beta}\to\frac{E^\beta}{\alpha^\beta}$ when $E\ll\alpha$. Therefore, for energies $E\ll E_0$, the slope of the spectrum is mainly determined by the parameter β and the amplitude of $D(E)$ depends on α at fixed β and $D_{\rm LIS}$. In this case, $D(E)$ has smaller amplitude for large α and vice-versa: small α indicates spectrum with larger amplitude and (2) at energies $E\gg\alpha$ Eq. (17) gives precisely the unmodulated spectrum $D_{\rm LIS}(E)$ as a function of the energy E.

Note that when the measurement spectrum $D(E)$ follows a power law, with some approximation, within the limits of the error of measurement, i.e. $D(E)\to D_{\rm LIS}(E)$ and K is unknown parameter, the following cases may be considered: (1) $\alpha\to 0$ and $\beta\to\infty$; (2) if $\alpha\gg E_0$, then the exponent β is very small positive

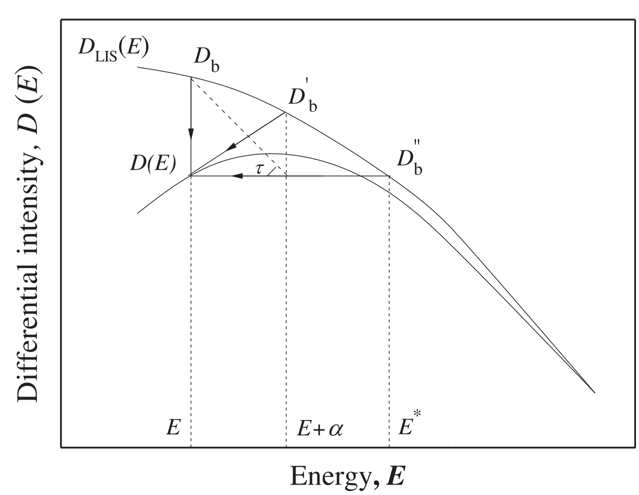

Fig. 1. Local interstellar spectrum $D_{\rm LIS}(E+E_0)$ and differential cosmic ray spectrum $D(E)$ in energy E. Graphical representation of the trigonometric dependence tan τ (Eq. (15)) using a logarithmic scale. The subscript b notes values $D_{\rm LIS}(E+E_0)$ for a given energy E on the outer boundary of the modulation region.

($D_{\rm LIS} < D(E)$ at high energy) or negative value ($D_{\rm LIS} > D(E)$ at high energy). The discussed situation can arise if (a) the spectrum is less sensitive to modulation, it is usually observed for alpha particles and (b) only one or two experimental points with approximately equal values are measured in an adiabatic range. Cases (1) and (2) are very sensitive to the initial parameter values. Sometimes case (2) is observed when the modulation still has some influence on the high energies of the spectrum (for example, during high solar activity) and the power index γ does not match the data for these energies quite precisely. Then, β has very small negative value.

For energies E below $\sim$0.4–0.6 GeV, an adiabatic cooling is essential and theoretical models predict the particle spectrum $D(E) \sim E$. This effect is demonstrated in the framework of the force–field model. Urch and Gleeson (1972a), assuming a mono-energetic galactic spectrum, have shown the energy spectrum $D(E)$ have a form $D(E) \sim E$ at $E < 200$ MeV. Experimentally, Kecskemety et al. (2001) found that in an energy range 10–100 MeV, the dominant part of the heliospheric modulation mechanisms provide spectrum slope $v_{\rm sl.} \approx 1.2$, during other time periods different processes lead to $v_{\rm sl.} \approx 0.9$, and sometimes weak ($v_{\rm sl.} \approx 0.5$) or stronger ($v_{\rm sl.} \approx 1.7$) modulation of galactic cosmic rays. Generally, the index of galactic CR spectrum in the adiabatic range below 300–400 MeV depends on many parameters of interplanetary medium, which determine the modulation: the diffusion coefficient, the spectrum of magnetic irregularities of the interplanetary magnetic field, etc. The different combinations from their values lead to spectrum slope with an index between 0.5 and $\sim$2.0 for the Earth (Kecskemety et al., 2001).

In the research range 0.02 MeV–100 GeV, the model equation Eq. (17) is a strictly convex function. But a strictly convex real function has a unique minimizer and the solution is reached when the gradient vanishes. Therefore, if we know γ we can obtain the range of parameter values α provided that the observed cosmic ray spectrum has maximum between 0.2 and 0.7 GeV and the optimal solutions of the parameter β, due to its physical interpretation are in the interval 0.6–1.9. The range of parameter values α is obtained by the expression

$$\alpha = \frac{\gamma E_{\rm max.}}{\gamma - \beta(1 + E_0/E_{\rm max.})} \tag{18}$$

$E_{\rm max.}$ is the energy at which the function $D(E)$ reaches the highest value.

4. Least square method for well-posed inverse problem

The function (Eq. (17)) relates the values of model parameters α, β and K (in case that K is given as an unknown one) to the results $D(E)$ of the measurements. In this case, the values of measured quantities are given and the theoretical relationship (Eq. (17)) is used in order for an information to be obtained on the values of the set of parameters. Therefore, we are solving an 'inverse problem.' Inverse problems can be *ill-* or *well-posed*. According to Hadamard (1923), a problem is called well-posed (or correctly-set) if

a. it has a solution,
b. the solution is unique,
c. the solution depends continuously on the data and the parameters (the stability criterion).

If one of these conditions is not satisfied, the problem is called ill-posed. Neither existence nor uniqueness of a solution to an inverse problem is guaranteed (Engl and Kügler, 2005).

The correct modeling of a physically relevant problem leads to a well-posed problem (Engl, 2005), so one can easily prove that the problem for the nonlinear Eq. (17) is well-posed. The meaning of (a) is clear: the existence of a solution for an inverse problem (Eq. (17)) is assured by physical and mathematical reasoning. Also, it can be shown that any small changes in parameters α, β (and K) will not affect the results $D(E)$, using Cauchy's definition for a limit of a function. Then, since the solution depends continuously on data and parameters, we do not have to worry about small errors in measurement producing large errors in our predictions. Because the model function (Eq. (17)) is strictly convex, it has a unique minimizer, i.e. a unique solution. (In Appendix A, it is shown that if α is constant, then Eq. (17) has a unique solution for $E \ll \alpha$.) With this we have proved that the inverse nonlinear problem (Eq. (17)) is well-posed. In this case, we can use Levenberg–Marquardt (LM) algorithm. This method is adjusted to well-posed problems, since its convergence analysis relies on the assumption that the derivative of the nonlinear operator is continuously invertible near the exact solution, which irrevocably fails to hold for ill-posed problems (Hanke, 2010; Kelley, 1999).

In practical inverse problems, due to errors in the measurements, one never has exact data. In this case, Engl and Kügler (2005) note that if the "deviation from the exact data is small, algorithm developed for well-posed problem can fail in case of a violation of the third Hadamard condition if it does not address the instability, since data as well as rounding-off the errors may then be amplified by an arbitrarily large factor". Fortunately, an LM is fairly stable method and it is widely recognized as the most efficient one in the sense of realization accuracy (Sperduti and Starita, 1993). An LM algorithm works well and the solution of the inverse problem is efficient and robust, but this method is very sensitive to the initial network weights and it does not consider outliers. In this case, more robust methods should be used.

An LM is a method used to solve non-linear least-squares problems that minimizes the fitting function $F(\boldsymbol{\theta}) = \Sigma_i [y(i) - f(x(i), \boldsymbol{\theta})]^2$. In this expression, $F(\boldsymbol{\theta})$ is the sum of squares (this function is called an object function), $\boldsymbol{\theta}$ is the vector of the fitted parameters, $y(i)$ is the observed values and $f(x(i), \boldsymbol{\theta})$ is the fitting function of a given quantity $x(i)$ and the fitted parameters $\boldsymbol{\theta}$, where i indexes the data points. If measurement points have different uncertainties, the objective function is computed by the weighted sum of squared errors, i.e. $F(\boldsymbol{\theta}) = \Sigma_i \omega_i [y(i) - f(x(i), \boldsymbol{\theta})]^2$, where ω_i is a weighting factor. Detailed information about the weighted least squares can be found in NIST/SEMATECH e-Handbook of Statistical Methods (2003).

If the errors in the measurements are statistically independent and Gaussian distributed, then the least square parameter estimation is identical to the maximum likelihood (ML) parameter estimation. In this case, a chi-square distribution can be used to test the goodness of the fit. An ML method is considered to have more desirable mathematical and optimality properties than the least squares method: (1) the average value of the parameter estimates is theoretically exactly equal to the population value (2) generates confidence bounds and since the estimator has the smallest variance it leads to the narrowest confidence interval (NIST/SEMATECH e-Handbook of Statistical Methods, 2003). Maximum likelihood statistic is preferred when the model is good and if (i) the measurement errors are normally distributed and either (ii) the fitting function is linear in its parameters or (iii) the sample size is large enough (Press et al., 1992). If these conditions are not fulfilled, then one can say that the covariance matrix is the "formal covariance matrix of the fit" (Press et al., 1992). In this sense, Press et al. (1992) write that "χ^2 minimization is a useful mean for estimating parameters even if the measurement errors are not normally distributed", $\chi^2 \equiv F(\boldsymbol{\theta})$.

The data for cosmic ray differential spectra often contains both systematic and random errors. Systematic errors always either overestimate or underestimate the results of measurements, while random errors both overestimate and underestimate the results. Correcting of the systematic errors may produce residuals that are normally distributed. Then, we consider the estimation of the model parameters using weighted least squares under the assumption of normality.

The reduced chi-squared (chi-squared divided by the number of degrees of freedom) χ_n^2 is a quantitative value that describes how well the model fits the data. When the value χ_n^2 is close to 1, it is an indicator of a good fit of the model solution to the data. A high value of χ_n^2 indicates an under-fitting. In the case of an under-fitting, the bias in the parametric estimation is generally substantial, while the variance is underestimated. Then, the chi-square statistic can be used to calculate a p-value by comparing the value of the statistic to a chi-square distribution, where p is the probability, under the null hypothesis. If $p \leq 0.05$, then the null hypothesis H_0 (the current model is correct) is rejected. Usually, p-value under 0.05 shows either lousy measurements or bad model. For one degree of freedom, the critical value associated with $p=0.05$ for χ_n^2 is 3.84. Chi-square values higher than this critical value are associated with a statistically low probability that H_0 is true. χ_n^2 too small indicates an over-fitting problem. Over-fitting produces parameter estimates that have large variances both in the parameter estimates and in the predicted values of the observed data. If variance and bias are optimized, the results will tend to have an appropriate fit. There are many ways of doing this and we will discuss them in a future work.

5. Fitting the model equation to experimental data and theoretical calculations

5.1. Fitting to the experimental data

The IMAX92 (Menn et al., 2000), CAPRICE94 (Boezio et al., 1999), AMS98 (Alcaraz et al., 2000a, b) and BESS (Shikaze et al., 2007) experimental spectra of galactic protons and BESS (Shikaze et al., 2007) measurements of alpha particles are fitted to the model spectrum (Eq. (17)) with an LIS, defined by Eq. (17a). In Tables 1 and 2 γ values are given (obtained from the measurement), the computed parameters α and β and the reduced chi-squared values χ_n^2 (or χ_n^2 per the degree of freedom) for the experiments IMAX92 (Menn et al., 2000), CAPRICE94 (Boezio et al., 1999), AMS98 (Alcaraz et al., 2000a), BESS 1997, 1998, 1999, 2000 and 2002 (Shikaze et al., 2007; Yamamoto, 2006) for protons. These parameters are shown in Table 3 for alpha particles for BESS (Shikaze et al., 2007) experimental spectra. The normalization constants K is chosen to match the modulated data above 20 GeV (/nucl), where the modulation effect is negligible. The calculation of the unknown parameters α and β is performed by Levenberg–Marquardt algorithm (Press et al., 1992). Since the influence of the systematic errors in measurements is not minimized, the values in Tables 1–3 are only approximate. However, for high signal-to-noise ratio if the shape of the model function has not been distorted, the main effect of systematic errors is to produce an offset, χ_0^2, in an experimentally determined $\chi_{\exp}^2$, i.e. $\chi_{\exp}^2 \approx \chi_{\text{rand}}^2(\theta_i)+\chi_0^2$, where θ_i is a varying parameter in a fit to real data (Booth and Hu, 2009).

The values of χ_n^2 are approximately between 1 and 2 for the experiments: IMAX92 (Menn et al., 2000), CAPRICE94 (Boezio et al., 1999), AMS98 (Alcaraz et al., 2000a), BESS 1998, 1999 and 2000 for protons and BESS 1999, 2000 and 2002 (Shikaze et al., 2007; Yamamoto, 2006) for alpha particles. The above estimates of χ_n^2 represent a good fit. Chi-square values for protons: BESS97 and BESS2002 indicate a more noticeable impact of the systematic errors in the measurements, which lead to an under-fitting. Table 3 shows that a reduced chi-square value is higher than critical 3.84 for BESS2002. It means either unmodeled noise or the estimated statistical errors are too small. In this case, we can put K in Eq. (17a) as an unknown parameter to achieve a better fit. Note that when the normalization constant K is taken as an unknown parameter, its value is determined as an optimal for the fit. The "best-fitting" parameter K determines the smaller value χ_n^2 than that which we would have if K was chosen to match the data at energies at which no modulation occurs. We remark though, that if there is no systematic error in the measurements, and K is considered an unknown parameter in Eq. (17), this can lead to an over-fitting. The problem of how to take into account experimental systematic errors during the inverse problem given by Eq. (17) will be discussed in our next paper. The reduced chi-square is <1.0, for experiments BESS97 and BESS98, for helium nuclei. It shows that the model over-fits data and the number of

Table 1
Fitting parameters K, α, β and χ_n^2 for protons for experiments IMAX 92 (Menn et al., 2000), CAPRCE 94 (Boezio et al., 1999) and AMS 98 (Alcaraz et al., 2000a).

Experiments	IMAX 92	CAPRICE94	AMS98
γ	2.61	2.73	2.76
K_{given}	8.030	11.000	15.345
α	0.779 ± 0.024	2.277 ± 0.008	1.183 ± 0.05
β	1.377 ± 0.027	1.206 ± 0.020	1.096 ± 0.028
χ_n^2	1.90	2.24	0.93

Table 2
Fitting parameters α, β and χ_n^2 for protons for experiments BESS97, BESS98, BESS99, BESS2000 and BESS2002 (Shikaze et al., 2007), $\gamma=2.732$ and $K=13.700$ (Yamamoto, 2006).

Experiments	BESS97	BESS98	BESS99	BESS2000	BESS2002
α	0.681 ± 0.016	0.845 ± 0.017	1.240 ± 0.023	2.707 ± 0.027	1.899 ± 0.012
β	1.205 ± 0.002	1.351 ± 0.018	1.242 ± 0.014	2.048 ± 0.011	2.221 ± 0.008
χ_n^2	3.06	2.40	2.50	2.26	5.26

Table 3
Fitting parameters α, β and χ_n^2 for helium nuclei for experiments BESS97, BESS98, BESS99, BESS2000 and BESS2002 (Shikaze et al., 2007), $\gamma=2.699$ and $K=0.706$ (Yamamoto, 2006).

Experiments	BESS97	BESS98	BESS99	BESS2000	BESS2002
α	5.5E-5 ± 4E-6	4.34E-4 ± 1.9E-5	0.186 ± 0.007	1.196 ± 0.019	0.934 ± 0.018
β	1749 ± 128	381 ± 17	1.577 ± 0.050	1.776 ± 0.017	1.753 ± 0.021
χ_n^2	0.83	0.53	0.90	1.14	1.20

adjustable parameters are higher. Then, a more robust approach is required.

In the computation, our goal was to extract β with value between 0.6 and 2, which has physical meaning for energies $E \ll \alpha$. Sometimes the model equation (Eq. (17)) fits the experimental measurements well without to reflect the physics of the data. Usually, it is observed when the experimental spectrum $D(E)$ follows a power law, with some approximation, within the limits of the error of measurement, i.e. $D(E) \to D_{\rm LIS}$ and $\alpha \to 0$ (see BESS97 and BESS98 in Table 3). Also, if K is an unknown parameter in Eq. (17) its value is necessary only for description of the measurement and it has no physical significance (similarly to the parameters α and β when $D(E) \to D_{\rm LIS}$). Generally, in these cases, calculated parameters have no physical meaning and they can be used only as mathematical estimates in the calculation to which the observed spectrum is applied.

5.2. Fitting to numerical solution of cosmic ray transport equation

The received proton spectra from a time dependent, two-dimensional stochastic simulation model of galactic CR propagation in the heliosphere (developed by Bobik et al. (2006)) for the Earth and the outer planets are fitted to Eq. (17). On the fit, the standard deviation for protons is 1% for the Earth and leads to 1.5% for Neptune when we use the power LIS (Eq. (17a)) in the model Eq. (17); $\gamma = 2.75$.

Exact values K, α and β are obtained for Earth and outer planets by the algorithm that combines the rapid local convergence of Newton method with a globally convergent method for non-linear systems of equations (Press et al., 1992). By inserting three characteristic spectral points (Ei, Di), one will get three unknowns K, α and β. The values K, α and β for each planet are given in Table 4. It is seen that the parameter α decreases with the distance, while β increases, i.e. the slope of modulated spectrum increase with the distance. The lower value of α is a result of the decreasing with the distance modulation, due to adiabatic losses. Actually, energy losses become smaller and the role of convection–diffusion process increase with an increasing radial distance. Thus, the slope of the low energy part of the GCR spectrum increase with the distance and in the outer heliosphere the differential spectrum has an exponential, rather than the power law form. In this case, Eq. (17) is not valid.

5.3. Fitting to FF solutions of cosmic ray transport equation. Relationship between the model parameters α and β and the modulation parameter Φ

Taking into account Eq. (9), the force–field relationship $f(r; P) = f_0\ (R, P_R)$ in terms of intensities $D(E)$ leads to

$$D(E) = (P/P_R)^2 D_{LIS}(E+\Phi) \qquad (19)$$

where $\Phi = (eZ/A)\varphi$ is the energy loss experienced incoming from distance R. The parameter Φ has the dimensions of energy. Thus

$$D(E) = D_{LIS}(E+\Phi)\frac{E(E+2E_0)}{(E+\Phi)(E+\Phi+2E_0)} \qquad (20)$$

Table 4
Computed parameter values K, α and β by fitting Bobiks' model (Bobik et al., 2006) for the Earth and outer planets to Eq. (16) using the power LIS (Eq. (16a)); $\gamma = 2.75$.

Planet	Earth	Jupiter	Saturn	Uranus	Neptune
K	16.539	15.162	14.560	14.565	15.187
α	1.740	1.346	0.976	0.532	0.255
β	1.040	1.087	1.158	1.267	1.469

The differential spectra $D(E)$ of galactic protons and alpha particles are calculated from Eq. (20) at given values of the modulation potential Φ. $D_{\rm LIS}(E)$ is an important factor in an FF approximation (Eq. (20)), because this model is dependent on the fixed shape of the LIS. Here, we use the LIS for protons given by Burger et al. (2000)

$$D_{LIS}(\mathrm{E}) = \frac{1.9\times10^4 P(E)^{-2.78}}{1+0.4866\,P(E)^{-2.51}} \qquad (21)$$

This spectrum lies between "empirically" derived and theoretically computed LIS in the range 1–10 GeV (Usoskin et al., 2005). The number ratio between alpha particles and protons in an LIS is assumed to be 0.05 (Usoskin et al., 2005).

The results from the differential energy spectra $D(E)$ of the primary protons for four values of modulation parameter: $\varphi = 400$, 550, 700 and 1000 MV are shown in Fig. 2. These results are given also for the helium nuclei in Fig. 3. The calculated spectra from Eq. (20) are approximated by model Eq. (17) using Burgers' LIS (Eq. (21)). The parameters α and β for different values Φ are given in Tables 5 and 6 for protons and alpha particles, respectively. On the fit, the standard deviations between the FF solutions (Eq. (20)) and the model (Eq. (17)) are ~(1–2)% for low

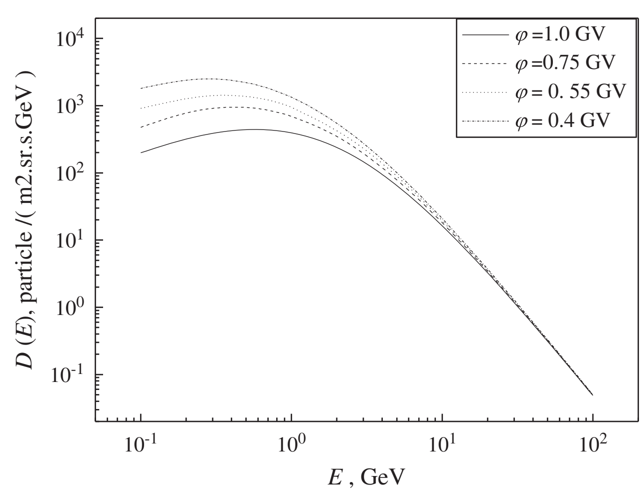

Fig. 2. Modeled $D(E)$ spectra of galactic protons with Burgers' LIS (Eq. (21)) for modulation levels $\varphi = 0.4$, 0.55, 0.7 and 1.0 GV.

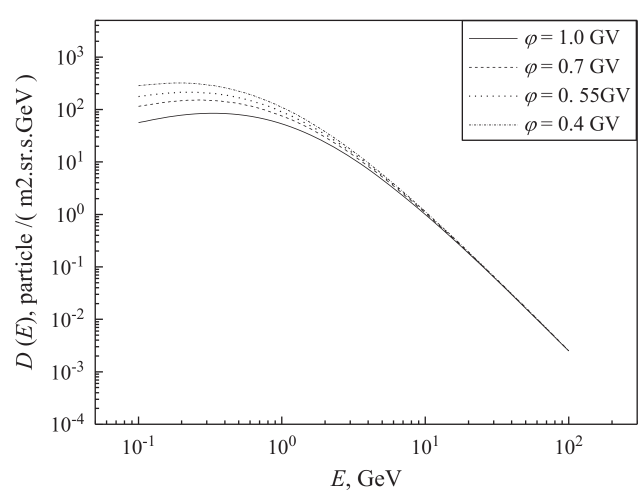

Fig. 3. Modeled $D(E)$ spectra of galactic helium nuclei with Burgers' LIS (Eq. (21)) for modulation levels $\varphi = 0.4$, 0.55, 0.7 and 1.0 GV. The lower depth of modulation for helium nuclei in comparison with protons is due to their higher rigidity P, for a given energy E.

and middle levels of modulation ($\varphi < 0.8$ GV) for protons and ~1% for all modulation levels for alpha particles. The standard deviations between Eqs. (17) and (20) are determined for $\chi_n^2 \le 1$. Therefore, Eq. (17) is good approximation to force–field solution describing the energy losses of CR in the inner heliosphere.

The expressions given by Eqs. (17) and (20) describe differential cosmic ray spectra when convection–diffusion effects can be neglected compared to adiabatic losses. These conditions are satisfied in the inner heliosphere, $r < 40$ AU and the model spectra (Eq. (17)) and (Eq. (20)) can be applied to describe galactic cosmic ray modulation during solar cycle from 1 to ~40 AU.

Substituting the expression for $D(E)$ from Eq. (17) in Eq. (20) and taking the natural logarithm of both sides, one gets

$$\ln\left[D_{LIS}(E+E_0)\left(1+\frac{\alpha}{E}\right)^{-\beta}\right] = \ln\left[D_{LIS}(E+\Phi)\frac{E(E+2E_0)}{(E+\Phi)(E+\Phi+2E_0)}\right] \quad (22)$$

For simplicity, we assume that the unmodulated intensity spectrum is a power law of energy. Then, from the above expression we get

$$\left(1+\frac{\alpha}{E}\right)^{\beta} = \left(\frac{E}{E+E_0}+\frac{\Phi}{E+E_0}\right)^{\gamma}\left(1+\frac{\Phi}{E}\right)\left(1+\frac{\Phi}{E+2E_0}\right) \quad (23)$$

For energies around and above 10 GeV, $E \gg E_0$ and the term in the first multiplier on the right $E/(E+E_0) \to 1$. Then, taking a logarithm of both sides of Eq. (23) we get

$$\beta \ln\left(1+\frac{\alpha}{E}\right) \approx \gamma \ln\left(1+\frac{\Phi}{E+E_0}\right) + \ln\left(1+\frac{\Phi}{E}\right) + \ln\left(1+\frac{\Phi}{E+2E_0}\right) \quad (24)$$

An FF theory is not precise during periods of high solar activity (Gleeson and Axford, 1968). Gleeson and Webb (1975) have obtained for the upper limit of the modulation potential the value $\varphi = 0.75$ GV for protons. That is why parameters α and β are given in Tables 5 and 6 from low to average modulation levels. It is seen that the value of α varies from 1 to ~3 for protons and alpha particles as α is always bigger than Φ. For energies $E \gg \alpha$ Eq. (24) can be approximated with

$$\beta\frac{\alpha}{E} \approx \gamma\frac{\Phi}{(E+E_0)} + \frac{\Phi}{E} + \frac{\Phi}{E+2E_0} \quad (25)$$

or

$$\alpha\beta \approx \gamma\frac{\Phi}{(1+E_0/E)} + \Phi + \frac{\Phi}{1+2E_0/E} \quad (26)$$

For $E \gg \alpha$, $E \gg E_0$ (Tables 5 and 6 show that $\alpha > E_0$), the term $(1+E_0/E) \to 1$ and $(1+2E_0/E) \to 1$. Thus, Eq. (26) becomes

$$\alpha\beta \approx (\gamma+2)\Phi = 3\Lambda\Phi, \quad (27)$$

where $\Lambda = \Lambda(\gamma) = (\gamma+2)/3$. Since the values of α in the FF model for low to average levels of modulation are of the order of the rest energy E_0, we can consider Eq. (25) approximately valid for energies about and above 15 GeV ($\alpha \approx 1.45$) at the low level of modulation ($\varphi = 300$ MV) and for energies ~30 GeV ($\alpha \approx 2.92$) and above them at an average level of modulation ($\varphi = 800$ MV) for protons. For $\varphi = 300$ MV, the modulation is ~10% at $E = 15$ GeV, and for $\varphi = 800$ MV, the modulation is ~13% at $E = 30$ GeV for protons. Therefore, the modulational parameters α, β and Φ still influence the formation of the spectrum for the dicussed energies. On the other hand, the standard deviation between FF model and the proposed approximate formula (Eq. (17)) is on the order of 1–2%, i.e. Eq. (17) almost exactly approximates an FF solution and the dependence between the modulation potential Φ and the parameters α and β will be slightly influenced by the standard deviations between the two spectrums.

For energies above 10–20 GeV, the value of γ is well determined theoretically (draft standard): $\gamma_p \approx 2.74$ for protons and $\gamma_\alpha \approx 2.68$ for alpha particles. However, sometimes the experimental measurements give some deviations from these γ values, but the differences are in very narrow ranges. Therefore, it can be assumed that $\Lambda(\gamma) \approx$ const. in Eq. (27). For theoretically derived power LIS with $\gamma_p = 2.74$ for protons and $\gamma_\alpha = 2.68$ for alpha particle calculations give us $\Lambda \approx 1.6$. Tables 5 and 6 confirm the dependence given by Eq. (27) with $\Lambda \approx 1.618$ when the calculated spectra from Eq. (20) are approximated by model Eq. (17) using Burgers' LIS (Eq. (21)) for protons and alpha particles.

6. Cosmic ray spectra approximation (CRSA) model in the heliosphere

CRSA model generalizes the differential galactic CR spectrum in the heliosphere for two extreme cases: convection–diffusion and energy losses. The formula expressed by Eq. (27) can be written as

$$\Phi \approx \frac{1}{\Lambda}\left(\frac{\alpha\beta}{3}\right) \quad or \quad \beta \approx \Lambda\left(\frac{3\Phi}{\alpha}\right) \quad (28)$$

Substitution of the expression for β from Eq. (28) in Eq. (17) results in the dependence

$$D(E) \approx D_{LIS}(E)\left(1+\frac{\alpha}{E}\right)^{-\Lambda\frac{3\Phi}{\alpha}} \quad (29)$$

Thus, Eq. (6) in terms of the differential intensity $D(E)$ and the approximation model (Eq. (29)) can be generalized in the form

$$D(E) = D_{LIS}(E)Q^{-N} \quad (30)$$

where Q and N are dimensionless qualities. Under condition of:

- energy losses: $Q = f(\alpha/E)$ is only dependent from the energy E. The exponent $N = f(\Phi/\alpha)$ is approximately equal to a constant for a given Φ.
- convection–diffusion: the exponent $N = M$ is dependent on an energy E, while Q is a mathematical constant, also known as Euler's number e. If the expression for the modulation function M (Eq. (7)) is substituted in Eq. (15a), a relationship between the force–field potential φ and the modulation function M is received

$$M = \frac{3\varphi}{k_2} \quad (31)$$

Table 5
Fitting parameters α and β for modulation levels Φ: 0.3, 0.4, 0.5, 0.6, 0.7 and 0.8 GeV for galactic protons.

Parameter	$\Phi=0.3$	$\Phi=0.4$	$\Phi=0.5$	$\Phi=0.6$	$\Phi=0.7$	$\Phi=0.8$
α	1.446600	1.766995	2.078808	2.372635	2.654086	2.924539
β	0.921775	1.049210	1.153789	1.244398	1.324410	1.396325

Table 6
Fitting parameters α and β for modulation levels $\Phi(=\varphi/2)$: 0.2, 0.25, 0.3, 0.35, 0.4, 0.45 and 0.5 GeV for galactic alpha particles.

Paramet.	$\Phi=0.2$	$\Phi=0.25$	$\Phi=0.3$	$\Phi=0.35$	$\Phi=0.4$	$\Phi=0.45$	$\Phi=0.5$
α	1.098153	1.273066	1.442468	1.606970	1.767006	1.922968	2.075121
β	0.759901	0.847191	0.922559	0.989216	1.049206	1.103899	1.154282

Then, Eq. (30) can be written as

$$D(E)=D_{LIS}(E)\left(1+\frac{(e-1)\delta^{\mu}E^{\nu}}{E}\right)^{-3\Phi(\Lambda/\alpha)^{\mu}(A/Z/k_2)^{\nu}} \tag{32}$$

Here, $\mu=(1-\nu)$ is an energy loss factor and ν is the convection–diffusion factor. The parameter $\alpha=(e-1)\delta$, where e is an irrational constant equal to 2.71828. The coefficient Λ is approximately equal to 1.618.

Eq. (32) is transformed in Eq. (29) or Eq. (17) by substituting $\mu=1$ ($\nu=0$). When ν is equal to 1 ($\mu=0$), the generalized model (Eq. (32)) has the form given with Eq. (6) in terms of the differential intensity $D(E)=vU(E)/4\pi$ taking into account Eq. (31).

7. Discussion and conclusion

In this work, a model approximation which generalizes the differential galactic cosmic ray spectrum in the heliosphere is proposed. The model parameterizes the spectrum at different physical conditions, including the most important effects controlling the CR intensity like convection–diffusion and energy losses. An FF formalism is a good approximation for galactic cosmic rays in the inner heliosphere, but its accuracy decreases towards the outer heliosphere. On the other hand, convection–diffusion approximation improves with the radial distance. The reason for the complementary behavior of these two approximations is that energy losses are relatively important in the inner heliosphere, but not in the outer heliosphere (Caballero-Lopez and Moraal, 2004). By a suitable choice of parameters, the proposed model (Eq. (32)) turns into two approximations: one close to the "force–field" model (describing the energy losses of CR in the inner heliosphere) and "convection–diffusion" equation (giving the reduction of CR intensity in the outer heliosphere).

Eq. (17) (or Eq. (32) with $\mu=1$) describes differential cosmic ray spectrum, when the convection–diffusion effects can be neglected compared to adiabatic losses. These conditions are satisfied in the inner heliosphere, $r<40$ AU and the model spectrum (Eq. (17)) can be applied to describe galactic cosmic ray modulation during the solar cycle from 1 to ~40 AU. A mathematical relation between parameters in the model approximation (Eq. (17)) and the modulation parameter Φ is derived. It is important to notice that modulation parameter Φ is not advisable (from physical considerations) on short time scales and during periods of an active Sun. The CRSA model has no such a problem. The modulation parameters α and β are not limited between the fixed values. It makes Eq. (17) (or Eq. (32) with $\mu=1$) more universal and applicable for different time scales and levels of solar activity. Tables 5 and 6 show that the value of α change in limits between 1 and 3, and β is about 1 when an FF solution (Eq. (20)) is approximated by Eq. (17). Therefore, from the values of α and β, derived from the modulation of the given experimental spectrum by Eq. (17), we can conclude whether an FF model is applicable towards the experimental data or not. For example, the very high values of α and β received for BESS2000 and BESS2002 (Tables 2 and 3) confirm the observed discrepancies between the measurement data and FF approximation (Shikaze et al., 2007). Therefore, the proposed model (Eq. (17)) is more flexible to the data fitting than an FF approximation.

The existing analytical, empirical and semi-empirical models parameterize the CR spectrum with a definite potential: heliospheric potential (O'Brien, 1971), deceleration potential (Badhwar and O'Neill, 1996) or modulation potential (Nymmik et al., 1996). The potential can be given in terms of sunspot numbers or in terms of the intensity of the galactic cosmic ray component measured with neutron monitoring (O'Brien, 1971). In our model, parameters are related to the solar-cycle-variation in the GCR intensity. The value of α increases and β changes very slowly in the transition from solar minimum to maximum. Generally, the parameter β is approximately equal to 1 around the Earth's orbit. The basic function of our model is to be used in atmospheric and heliospheric computations during the solar cycle. However, the model of Nymmik et al. (1992, 1996) or CREME96 (Tylka et al., 1997)[1] relates the solar-cycle variation in cosmic ray intensity to the observed sunspot number. Because there are detailed sunspot number predictions, these models can be used to predict galactic cosmic ray intensity also. The dependence of α and β on the solar activity parameters (for example, sunspot numbers) will be discussed in a next paper.

The differential $D(E)$ spectrum (Eq. (17)) of galactic CR is used for computation of the electron production rate profiles in the atmospheres and ionospheres for lower, middle and high latitudes in Velinov et al. (2004) and Velinov and Mateev (2008). That is why it is important the expression for cosmic ray differential spectrum to be presented in a simple form as Eq. (17) convenient at ionization computations and this region of applicability to exceed some limitations of analytically derived model approximations to the transport equation.

Acknowledgements

This work was supported by Bulgarian Ministry of Education, Youth and Science (grant no. DDVU-02/104/2010).

Appendix A

When $E\ll\alpha$ Eq. (17) has a form

$$\frac{D(E)}{D_{LIS}(E+E_0)}\approx\frac{E^{\beta}}{\alpha^{\beta}} \tag{A.1}$$

We construct two solutions with given energy E, T_1 and T_2, each of which satisfies Eq. (17) for boundary condition $E\ll\alpha$

$$T_1=\frac{E^{\beta_1}}{{\alpha_1}^{\beta_1}} \tag{A.2}$$

$$T_2=\frac{E^{\beta_2}}{{\alpha_2}^{\beta_2}} \tag{A.3}$$

With the new variables, $\zeta=T_1-T_2$, the governing equation is

$$\zeta=\frac{E^{\beta_1}}{{\alpha_1}^{\beta_1}}-\frac{E^{\beta_2}}{{\alpha_2}^{\beta_2}} \tag{A.4}$$

Taking into account that Eq. (A.1) has a unique solution, i.e. T_1–$T_2=0$ Eq. (A.4) leads to

$$E^{(\beta_1-\beta_2)}=\frac{(\alpha_1)^{\beta_1}}{(\alpha_2)^{\beta_2}} \tag{A.5}$$

If α is an independent parameter from the energy E then $\beta_1=\beta_2$ and $\alpha_1=\alpha_2$. The converse does not hold i.e. if $\alpha_1=\alpha_2$ and $\beta_1=\beta_2$, the quantity α is not necessarily to be an independent parameter from the energy E in Eq. (A.5).

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[1] GCR spectra in CREME96 (Tylka et al., 1997) are based on the model of Nymmik et al. (1992). Nymmik et al. (1992, 1996) model quantitatively describes spectra of electrons and ions with atomic numbers from 1 to 28. CREME96 extends this model through uranium (atomic number $Z=92$).

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