Scalable quantum search using trapped ions

S. S. Ivanov,[1,2] P. A. Ivanov,[1,3] I. E. Linington,[1,4,*] and N. V. Vitanov[1,5]

[1]*Department of Physics, Sofia University, James Bourchier 5 Boulevard, BG-1164 Sofia, Bulgaria*

[2]*The School of Physics and Astronomy, University of Leeds, Leeds LS2 9JT, United Kingdom*

[3]*Institut für Quanteninformationsverarbeitung, Universität Ulm, Albert-Einstein-Allee 11, D-89081 Ulm, Germany*

[4]*Department of Physics and Astronomy, University of Sussex, Falmer, Brighton BN1 9QH, United Kingdom*

[5]*Institute of Solid State Physics, Bulgarian Academy of Sciences, Tsarigradsko Chaussée 72, BG-1784 Sofia, Bulgaria*

(Received 11 January 2010; published 29 April 2010)

We propose a scalable implementation of Grover's quantum search algorithm in a trapped-ion quantum information processor. The system is initialized in an entangled Dicke state by using adiabatic techniques. The inversion-about-average and oracle operators take the form of *single* off-resonant laser pulses. This is made possible by utilizing the physical symmetries of the trapped-ion linear crystal. The physical realization of the algorithm represents a dramatic simplification: each logical iteration (oracle and inversion about average) requires only *two* physical interaction steps, in contrast to the large number of concatenated gates required by previous approaches. This not only facilitates the implementation but also increases the overall fidelity of the algorithm.

DOI: 10.1103/PhysRevA.81.042328 PACS number(s): 03.67.Ac, 03.67.Lx, 03.67.Bg, 42.50.Dv

I. INTRODUCTION

One of the most celebrated applications of quantum information processing is Grover's quantum search algorithm, which allows an initially unknown element to be determined from $\mathcal{N}$ equally likely possibilities in $O(\sqrt{\mathcal{N}})$ queries [1]. This outperforms the optimum classical strategy (a random "trial and error" of elements), which requires $O(\mathcal{N})$ steps on average. In addition to providing a speed-up of the unstructured search problem, Grover's algorithm can also be adapted to look for solutions to a range of mathematically difficult problems that have structure, by nesting one quantum search inside another [2]. As with other applications of quantum computing, the benefits of a quantum over a classical approach increase with the size of the database. Indeed, it has been suggested that the *primary resource* for quantum computation is a Hilbert-space dimension, which grows exponentially with the available physical resources [3].

Proof-of-principle quantum search has been successfully demonstrated in nuclear magnetic resonance [4–6], linear optical [7,8] and trapped-ion systems [9], as well as with individual Rydberg atoms [10] and in classical optics [11]. Of these, only the trapped-ion platform possesses a fully scalable Hilbert space and in this sense it is realistically the only candidate for performing a practically useful quantum search. We note, however, that while the trapped-ion system is scalable [12], the largest dimensional quantum search so far performed with trapped ions was for a database size of $\mathcal{N} = 4$ [9]. Extending the approach of Ref. [9] to a large number of ions is highly demanding, since it requires the ability to construct, with very high fidelity, a great number of multiply conditional gates. For two ions, as in Ref. [9], each oracle query amounts to a controlled phase gate between the ions. When N ions are involved, however, the oracle operator becomes a gate which is multiply conditional upon the internal state of *all N ions*. Although such a multiply conditional gate can be decomposed efficiently using a series of one- and two-ion gates [13], in practice such a synthesis becomes a daunting task, even for a moderate number of ions N, since the number of the elementary gates grows exponentially with N. For example, for a four-qubit register, the oracle may be constructed using 13 two-qubit conditional gates [13], each requiring several consecutive physical interactions, which is beyond the capabilities of current experiments [14–16]. The above example makes it apparent that there is a clear distinction between an approach which is *formally* scalable and an approach which is *realistically* scalable using current technology.

It is highly desirable, therefore, to find new ways to perform quantum search in a scalable system, which does not require the implementation of an exorbitant number of one- and two-qubit gates. In this paper we propose an approach to perform Grover's search algorithm. The linear crystal of N ions is prepared in a symmetric Dicke state with $N/2$ excitations, $|W^N_{N/2}\rangle$; hence, the register dimension scales *exponentially* with the number of ions. The implementation of the algorithm involves a series of red-sideband laser pulses addressing all, or half, of the ions in the trap. Thus, each of the inversion-about-average operator and the oracle query can be produced in a *single* interaction step. Consequently, the total number of *physical* steps is the same as the number of algorithmic steps.

The implementation of Grover's search algorithm proposed below follows earlier proposals, wherein the database scales linearly [17] (see also [18]) and quadratically [19] with the number of ions, N. The database size here, which is $\mathcal{N} \sim 2^N\sqrt{2/(\pi N)}$, scales exponentially with N and it is therefore of fully quantum nature. The reduction of the full Hilbert space of dimension 2^N by a factor of $\sqrt{N}$ is compensated by the absence of any ancilla qubits or the need of error correction because of the ultracompact implementation, in which each logical mathematical step is realized with a single physical interaction (laser pulse).

The remainder of this paper is organized as follows. A brief review of Grover's algorithm for a quantum-mechanical speed-up of the unstructured search problem is given in Sec. II. Particular attention is given to the ideas of amplitude amplification and generalized reflections, which lie at the heart of the

*Deceased.

search algorithm. Section III introduces a model Hamiltonian describing a single laser pulse illuminating a chain of trapped ions, and a convenient Hilbert-space factorization, with which to describe the dynamics. By making use of the simplified dynamics in this factorized basis, we construct recipes for the approximate synthesis of the two reflections required for a quantum search, and examples of numerical simulations are presented in Sec. IV. In Sec. V we summarize our findings.

II. GROVER'S SEARCH ALGORITHM

Grover's algorithm provides a method for solving the unstructured search problem, which can be stated as follows: given a collection of database elements $x = 1,2,\ldots,\mathcal{N}$, and an *oracle function* $f(x)$ that acts differently on one *marked* element s to all others,

$$f(x) = \begin{cases} 1, & x = s, \\ 0, & x \neq s, \end{cases} \tag{1}$$

find the marked element in as few calls to $f(x)$ as possible [1]. If the database is encoded in a physical system that behaves classically, then each oracle query can only act on a single database element. In this case, the optimal search strategy is simply a random selection of elements; on average, it would be necessary to make approximately $\mathcal{N}/2$ calls to the oracle before the marked element s is located. The idea underlying Grover's algorithm is to encode the database in a physical system that behaves quantum-mechanically. Therefore, each possible search outcome is represented as a basis vector $|x\rangle$ in an $\mathcal{N}$-dimensional Hilbert space; correspondingly, the marked element is encoded by a *marked state* $|s\rangle$. Hence, one can apply unitary operations (involving the oracle function) to *superpositions* of the different search outcomes. It is thus possible to amplify the amplitude of the marked state $|s\rangle$ using constructive interference, while attenuating all other amplitudes, and to locate the marked element in $O(\sqrt{\mathcal{N}})$ steps. Before the execution of the algorithm, the quantum register is prepared in an *equal* superposition of all basis elements,

$$|W\rangle = \frac{1}{\sqrt{\mathcal{N}}} \sum_{x=1}^{\mathcal{N}} |x\rangle. \tag{2}$$

Central to the operation of the quantum search algorithm is the idea of generalized complex reflections, known in the computer science literature as *Householder reflections* [20]:

$$\hat{M}_\psi(\phi) = \mathbf{1} + (e^{i\phi} - 1)|\psi\rangle\langle\psi|. \tag{3}$$

When the phase ϕ is set equal to π, the effect of $\hat{M}_\psi(\phi)$ on any vector is to invert the sign of the component of this vector along $|\psi\rangle$ while leaving all other components unchanged, which amounts to a reflection with respect to an $(\mathcal{N}-1)$-dimensional plane orthogonal to $|\psi\rangle$. Allowing ϕ to take arbitrary values extends the concept of reflection by imprinting an arbitrary phase onto the component along $|\psi\rangle$, rather than a simple sign inversion. Householder reflections are widely used in classical data analysis and also constitute a powerful tool for coherent manipulation of quantum systems [21–25]. The core component of Grover's algorithm is a pair of coupled Householder reflections, which together form a single *Grover operator* $\hat{G}$:

$$\hat{G} = \hat{M}_W(\varphi_W)\,\hat{M}_s(\varphi_s). \tag{4}$$

According to standard nomenclature, the operator $\hat{M}_s(\varphi_s)$ is referred to as the oracle query, while $\hat{M}_W(\varphi_W)$ is known as the inversion-about-average operator.

We note that with the initial state given in Eq. (2), and during successive applications of the operator $\hat{G}$, the state vector for the system begins and remains in the two-dimensional subspace defined by the nonorthogonal states $|s\rangle$ and $|W\rangle$. Each application of $\hat{G}$ amplifies the marked state population until it reaches a maximum value close to unity after n_G iterations, at which point the search result can be read out by a measurement in the computational basis.

The problem of how to optimize the quantum search routine by allowing arbitrary φ_W and φ_s has been studied extensively [26–28]. It is found that the maximum possible amplitude amplification per step of the marked state arises when the phases φ_W and φ_s are both set to π (as in Grover's original proposal [1]). The corresponding minimum number of search steps, $n_G^{\min}$, is given by

$$n_G^{\min} = \left[\frac{\pi}{2\arcsin(2\sqrt{\mathcal{N}-1}/\mathcal{N})}\right] \xrightarrow{\mathcal{N}\gg 1} \left[\frac{\pi\sqrt{\mathcal{N}}}{4}\right], \tag{5}$$

where $[n]$ denotes the integer part of n. However, this choice of phases is not unique. For large $\mathcal{N}$, as long as the *phase matching* condition $\varphi_W = \varphi_s = \varphi$ is satisfied [27], a high-fidelity search can be performed for any value of φ in the range $0 < \varphi \leqslant \pi$, and for certain values of φ, a deterministic quantum search is possible [26].

In the following two sections, we describe how to perform a physically efficient quantum search using the dynamic symmetries in a system of trapped ions.

III. ION-TRAP IMPLEMENTATION

A. Model Hamiltonian

We consider N ions confined in a linear Paul trap, which are cooled to their vibrational ground states. Each has two relevant internal states $|0\rangle$ and $|1\rangle$, with respective transition frequency ω_0. The linear ion crystal interacts with a laser pulse with frequency $\omega_L = \omega_0 - \omega_{tr} - \delta$, where ω_{tr} is the axial trap frequency and δ is the laser detuning from the first red-sideband resonance. We assume that the phonon spectrum can be resolved sufficiently well that only the center-of-mass mode is excited by this interaction and that other vibrational modes can safely be neglected. In the Lamb-Dicke limit, and after making the optical and vibrational rotating-wave approximations, the interaction Hamiltonian is [29]

$$\hat{H}_I(t) = \hbar \sum_{k=1}^{N} \left[g(t)(\sigma_k^+ \hat{a} + \sigma_k^- \hat{a}^\dagger) + \frac{\delta}{2}\sigma_k^{(z)} \right], \tag{6}$$

where $\sigma_k^+ = |1_k\rangle\langle 0_k|$ and $\sigma_k^- = |0_k\rangle\langle 1_k|$ are the raising and lowering operators for the internal states of the kth ion, $\sigma_k^{(z)}$ is the Pauli spin matrix, and $\hat{a}^\dagger$ and $\hat{a}$ are respectively the

creation and annihilation operators of center-of-mass phonons. The coupling between the internal and motional degrees of freedom is $g(t) = \eta\Omega(t)/\sqrt{N}$, where $\eta = \sqrt{\hbar|\mathbf{k}|^2/2M\omega_{\rm tr}}$ is the single-ion Lamb-Dicke parameter, with $\mathbf{k}$ being the laser wave vector and M the mass of the ion. The function $\Omega(t)$ is the real-valued time-dependent Rabi frequency.

Hamiltonian (6) conserves the total number of excitations (n_i ionic plus n_p motional), which, in the scheme we propose, is half the number of ions: $N/2$ (with N even), $n_i + n_p = N/2$. The energy pattern splits into manifolds corresponding to n_i ionic and $n_p = N/2 - n_i$ motional excitations. Each manifold is $C^N_{n_i}$-fold degenerate, where $C^N_{n_i} = N!/n_i!(N-n_i)!$. It is readily verified that the dimension of the $n_i = N/2$ manifold, $\mathcal{D}$, grows *exponentially* with N; indeed, for large N we have

$$\mathcal{N} \equiv C^N_{N/2} \sim \frac{2^N}{\sqrt{\pi N/2}}\left[1 - \frac{1}{4N} + O(N^{-2})\right]. \qquad (7)$$

We use the subspace of the overall Hilbert space, which spans the manifold $\mathcal{D}$, to represent the state of the register in Sec. IV; $\mathcal{D}$ is the set of states which encode the database of $\mathcal{N}$ elements.

B. Hilbert-space factorization

It is possible to move to a new basis in which the Hilbert space is factorized into a collection of noninteracting chains of states. The new states we shall call MS states, since they can be obtained by the multilevel Morris-Shore (MS) factorization [30].

To determine the MS states, first we rewrite Hamiltonian (6) in terms of the total ionic pseudospin:

$$\hat{H}_I(t) = \hbar g(t)(\hat{a}\hat{J}_+ + \hat{a}^\dagger \hat{J}_-) + \hbar\delta\hat{J}_z, \qquad (8)$$

where $\hat{J}_\pm = \sum_{k=1}^N \sigma_k^\pm$ and $\hat{J}_z = \frac{1}{2}\sum_{k=1}^N \sigma_k^{(z)}$. The MS basis consists of the set of the eigenvectors of the two commuting operators $\hat{J}^2$ and $\hat{J}_z$, where $\hat{J}^2 = \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) + \hat{J}_z^2$. Therefore, each MS state is assigned two quantum numbers, respectively j and m_j. Since $\hat{J}^2$ commutes also with Hamiltonian (8), the Hilbert space factorizes into a set of decoupled chains with different values of j.

The MS states are sketched in Fig. 1 for six ions with three excitations. Because the laser pulse couples equally to every ion in the trap, the longest chain is comprised of the symmetric Dicke states $|W^N_{n_i}\rangle$, wherein a given number of ionic excitations n_i is shared evenly among all the ions in the trap [31]. Each chain is assigned different j and is comprised of states with different m_j, which varies from 0 to j, corresponding to the number of excited ions. If n_i of the ions are excited, then

$$m_j = n_i - \frac{N}{2} = -n_p. \qquad (9)$$

Since the total number of excitations is half the number of ions, then from Eq. (9) it follows for each chain that m_j has a maximum value of 0 (i.e., states with $m_j > 0$ are inaccessible). By analogy with the traditional angular momentum operators, it follows that the number of states in a chain is equal to $j+1$ (j of the states are not accessible), from which j can be inferred. The longest chain corresponds to $j = N/2$, the next longest to $j = N/2 - 1$, and so on, and overall there are $[N/2+1]$ chains of different length in the factorized coupling scheme.

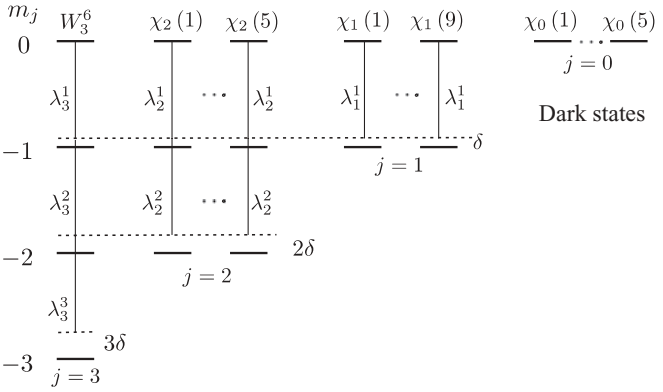

FIG. 1. The MS basis states for $N = 6$ ions with three excitations form a series of independent chainwise linkages. Since the total number of excitations is half the number of ions, then from Eq. (9) it follows that states with $m_j > 0$ are inaccessible and are therefore not shown. The states that make up the longest ladder are all symmetric Dicke states [31]. The number of motional n_p and ionic n_i excitations for each level can be inferred from m_j [Eq. (9)].

For the following analysis, it is necessary to go further and calculate the couplings in the MS basis. The coupling between the neighbors $|j,m_j\rangle$ and $|j,m_j-1\rangle$ follows immediately from the matrix elements of the operators $\hat{a}^\dagger$ and $\hat{J}_-$:

$$\begin{aligned}\lambda_j^{-m_j}(t) &= g(t)\sqrt{n_p(j+m_j)(j-m_j+1)} \\ &= g(t)\sqrt{n_p(j+n_i-N/2)(j+N/2+1-n_i)}.\end{aligned} \qquad (10)$$

As illustrated in Fig. 1, there can be many different degenerate MS states with the same values of j and m_j. We label the states with a given $j \neq N/2$ and $m_j = 0$ with $|\chi_j(k)\rangle$, where $k = 1,\ldots,N_j$, with $N_j = C^N_{N/2-j} - C^N_{N/2-j-1}$.

IV. IMPLEMENTATION OF GROVER'S ALGORITHM

The manifold $\mathcal{D}$, which encodes the database elements, consists of the states, for which the pseudoangular momentum projection is $m_j = 0$ [Eq. (9)]. This requires that the total number of ions N is even. Half of these are in state $|1\rangle$, while the other half are in state $|0\rangle$ ($n_i = N/2$). The number of elements $\mathcal{N}$ in the database therefore scales exponentially with the number of ions N [Eq. (7)]. The elements $|x\rangle$ in Eq. (2), which belong to $\mathcal{D}$, can now be written as

$$|x\rangle = P_x|1_1\cdots 1_{N/2}0_{N/2+1}\cdots 0_N\rangle, \qquad (11)$$

where the subscript x runs over all distinct permutations P_x of the ions' internal states. The initial state $|W\rangle$ is thus a symmetric Dicke state $|W^N_{N/2}\rangle$ of N ions sharing $N/2$ excitations,

$$\left|W^N_{N/2}\right\rangle = \frac{1}{\sqrt{C^N_{N/2}}}\sum_x P_x|1_1\cdots 1_{N/2}0_{N/2+1}\cdots 0_N\rangle. \qquad (12)$$

Our proposed experimental procedure consists of four operations. (i) The ions are first initialized in the entangled Dicke state [Eq. (12)]. This may be achieved using adiabatic passage techniques, involving either a pair of chirped laser pulses [31,32] or two pairs of delayed but overlapping laser

pulses [33], and using global addressing. (ii) Synthesis of the inversion-about-average operator is appealingly compact: $\hat{M}_W$ is a single red-sideband off-resonant laser pulse addressing all ions in the linear chain. (iii) The oracle query $\hat{M}_s$ is also a single red-sideband laser pulse, applied on half of the ions. This could be performed by transferring the other half to an electronic state, which does not interact with the laser, as explained in Ref. [34]. After an appropriate number of iterations [Eq. (5)], the system evolves into the marked state $|s\rangle$, which can be identified by performing (iv) a fluorescence measurement on the entire chain.

A. Synthesis of the inversion-about-average operator

In most existing proposals for implementing Grover's search algorithm using trapped ions, the generation of $\hat{M}_W$ requires a large number of concatenated physical interaction steps, even for moderate register size $\mathcal{N}$. However, by restricting the dynamics to the subspace of the overall Hilbert space in which only half of the ions are excited, it becomes possible for this operator to synthesize in only a *single* interaction step. This simplification is achieved by taking advantage of the fact that both Hamiltonian (8) and the state $|W^N_{N/2}\rangle$ are symmetric under exchange of any two ions.

The energies of the MS states do not cross in time so that in the limit $\delta \gg 1/T$ the transitions between the MS states vanish due to the effect of adiabatic complete population return [35]. Each of the MS states acquires a phase shift φ_j (where the index j corresponds to the eigenvalue of $\hat{J}^2$) and the unitary propagator within the Dicke manifold $\mathcal{D}$ is a product of $C^N_{N/2-1}$ coupled reflections [24]

$$\hat{U}_W = \hat{M}_W(\varphi_W) \prod_{j=1}^{N/2-1} \prod_{k=1}^{N_j} \hat{M}_{\chi_j(k)}(\varphi_j), \tag{13}$$

with $N_j = C^N_{N/2-j} - C^N_{N/2-j-1}$ and $\varphi_W = \varphi_{N/2}$. For a given value of the coupling, the detuning may be adjusted in order to control the phases φ_j. Ideally, these interaction parameters would be chosen such that $\varphi_W \neq 0$ (e.g., $\varphi_W = \pi$), while $\varphi_{j\neq N/2} = 0$ as this would result in $\hat{U}_W$ being identical to the inversion-about-average operator $\hat{M}_W(\varphi_W)$.

B. Synthesis of the oracle operator

The effect of each oracle query $\hat{M}_s(\phi_s)$ is to imprint a phase shift ϕ_s on the marked state $|s\rangle$, while leaving all other computational basis states unchanged. In general this can be achieved by a multiply conditional phase gate upon the internal state of all N ions in the trap. When more than a few ions are involved, this becomes a prohibitively complicated operation, which generally requires an immense number of one- and two-qubit gates [13]. However, since we work not in the whole Hilbert space but rather in the manifold $\mathcal{D}$, the oracle operator can be implemented in a much simpler fashion—by a *single* red-sideband laser pulse, addressing uniformly those $N/2$ ions in the trap which share the excitation of the marked state.

Let us consider an example, when the marked state is $|s\rangle = |111000\rangle$. Then we address the first three ions. Since the initial state $|W^N_{N/2}\rangle$ and the interactions with the laser are symmetric under exchange of the first three ions, and the last three, as well, during the execution of the algorithm, the state of the system is a linear combination of the states $|\Phi_k\rangle = |W^{N/2}_{N/2-k}\rangle|W^{N/2}_k\rangle$, $k = 0, \ldots, N/2$; the marked state is $|s\rangle = |\Phi_0\rangle$. Because the Hamiltonian, which describes the oracle call, does not change the total number of excitations of the first (and the last) three ions (i.e., it does not drive the system between $|\Phi_{k_1}\rangle$ and $|\Phi_{k_2}\rangle$ for $k_1 \neq k_2$), in the adiabatic limit the states $|\Phi_k\rangle$ acquire only phase shifts ϕ_k. Therefore, the propagator in the manifold $\mathcal{D}$ is formally described by the action of $N/2$ coupled reflections,

$$\hat{U}_s = \hat{M}_s(\phi_s) \prod_{k=1}^{N/2-1} \hat{M}_{\Phi_k}(\phi_k). \tag{14}$$

Hence, as in the case of the inversion-about-average operator, we need to control $N/2$ phases. The interaction parameters should be chosen such that $\phi_s = \varphi_W$, while $\phi_{k\neq N/2} = 0$, as this would result in $\hat{U}_s$ being identical to the oracle operator $\hat{M}_s(\phi_s)$.

C. Phase conditions

The phase conditions for the propagators $\hat{U}_W$ and $\hat{U}_s$, derived above, cannot be satisfied exactly, since the phases φ_k and ϕ_k are overly commensurate. However, one can still perform the algorithm with sufficiently high fidelity:

$$P = 1 - 2\,|\mathrm{Re}\langle f|\Delta f\rangle|\,. \tag{15}$$

Here $|f\rangle$ is the state of the system after the completion of the algorithm and $|\Delta f\rangle$ is its deviation due to the phases deviation. Fortunately, a Fourier expansion reveals that phase deviation of all odd orders leads to the purely imaginary bracket $\langle f|\Delta f\rangle$ and, hence, does not contribute to Eq. (15). As a result, the algorithm is less sensitive to phase deviations since only those of second order have a leading effect on the final state populations.

D. Practical considerations

Below we briefly discuss different experimental issues which are usually encountered in practice. Ideally we need uniform addressing of the ion chain to initialize the system and to perform the search algorithm. To estimate the error due to a possible inhomogeneous addressing, we assume a Gaussian spatial distribution of the laser intensity. Then the first and the last ions experience a fraction $1 - \xi$ of the intensity of the field in the center of the chain. According to calculations, for a deviation of $\xi = 0.01$, the fidelity decreases to about 0.96 for $N = 6$ ions. A possible way to achieve a more robust implementation is to use composite pulses, which have been used successfully with trapped ions [36].

Another potentially detrimental effect could be caused by the nonresonant transitions to vibrational modes other than the center-of-mass mode. These have a negligible effect if the detuning is much less than the trap frequency ω_z and the Rabi frequency is less than the trap frequency ω_z. For the implementation of the oracle and the inversion-about-average operators for $N = 8$ ions, we can use a laser pulse with width $T = 60\ \mu$s. Hence, the Rabi frequencies and the detunings are approximately $\Omega_o/2\pi = 810$ kHz and $\delta_o/2\pi = 114$ kHz for the oracle and $\Omega_r/2\pi = 915$ kHz and $\delta_r/2\pi = 56$ kHz for the inversion-about-average operator. The ac Stark shift due

to the presence of the off-resonant carrier transition gives rise to a phase ϕ for each qubit in state 1 and a phase $-\phi$ for each qubit in state 0. Therefore, the total phase for each state of the Dicke manifold is zero because it consists of an equal number of qubits in states 0 and 1. Moreover, the off-resonant carrier transitions can be reduced by using pulse shapes which have no spectral Fourier component at the carrier transition [36].

A major experimental obstacle is the motional decoherence, especially if the interaction takes a long time. The total interaction time for the implementation of the Grover algorithm is $T_G = N_G(3T_o + 3T_r)$, where T_o and T_r are the laser pulse widths for the oracle and the inversion-about-average, respectively, and the factor of 3 ensures that the pulses are well separated from each other (this factor may be reduced with appropriate arrangements). For $N = 8$ ions, the oracle needs half the coupling compared to the inversion-about-average; hence, we take $T_o = T_r/2$. Therefore, the total interaction time for the algorithm is $T_G = 27T_r$. Hence, for pulse width $T_r = 60$ μs, we have $T_G = 1.6$ ms. For a trap frequency $\omega_z = 2\pi \times 4$ MHz and a heating rate for a single $^{40}Ca^+$ ion of approximately 5 phonons/s [37], we predict heating of about 0.06 phonons in total.

Any fluctuations of the applied magnetic fields cause a collective dephasing of the qubit states which limits the storage time. An advantage of our method is that the quantum information is encoded in the Dicke manifold which is a decoherence-free subspace, because it consists of states with half of the ions in the excited state and half in the ground state. Hence, quantum information is protected from the induced collective dephasing, which is the limit to quantum memory, as explained in Ref. [38].

E. Numerical demonstration

As a demonstration, we have solved the Schrödinger equation numerically for a Gaussian pulse shape and a constant detuning δ. Sample results are shown in Fig. 2 and Table I for $N = 6, 8$, and 10 ions with 3, 4, and 5 excitations, respectively, which imply databases of $\mathcal{N} = 20$, 70, and 252 elements. The fidelity plotted on the vertical axis is the time-dependent population of the marked state. The system of ions is assumed to be prepared initially in the Dicke superposition $|W^N_{N/2}\rangle$ of ionic collective states, each of which contains exactly $N/2$ ion qubits in state $|1\rangle$ and $N/2$ in state $|0\rangle$. Each Grover iteration consists of a phase shift for the marked state (oracle call), which amounts a single red-sideband laser pulse, addressing uniformly those $N/2$ ions in the chain which share the excitation of the marked state, followed by a single red-sideband pulse, which addresses uniformly the entire ion chain (the inversion-about-average operator). The number of steps, for which the algorithm singles out the marked item with a probability of around 99%, is respectively $n_g = 3$, 6, and 12 as predicted by Eq. (5).

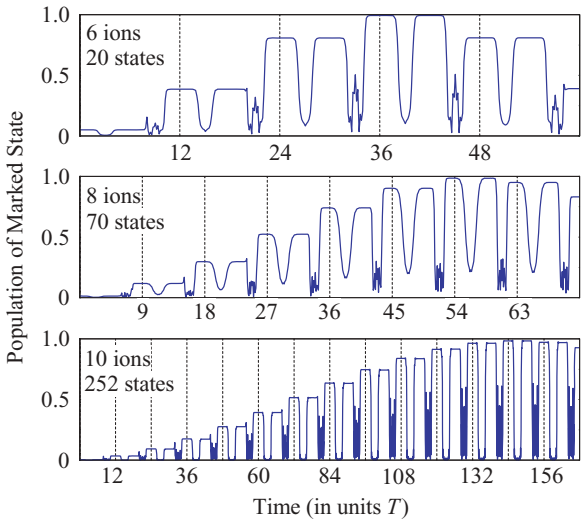

FIG. 2. (Color online) Simulation of the Grover search algorithm with N ions and $N/2$ excitations (register dimension $C^N_{N/2}$), for $N = 6$ (upper panel), $N = 8$ (middle panel), and $N = 10$ (lower panel). The system of ions is assumed to be initialized in the symmetric Dicke state $|W^N_{N/2}\rangle$. The laser pulses have a Gaussian shape, $g(t) = g_0 e^{-t^2/T^2}$. The thin vertical lines display the times when each logical mathematical step (an implementation of the Grover operator) is completed. The computed numerically scaled detunings and peak Rabi frequencies for the oracle and the inversion-about-average operators, respectively, are given in Table I. The marked-state population (the fidelity) of around 99% is obtained after $n_g = 3$, 6, and 12 steps, respectively, in exact agreement with Grover's value [Eq. (5)]. The numerical simulation includes all off-resonant transitions to states with $m_j \neq 0$.

V. CONCLUSIONS

Despite the intense flurry of theoretical and experimental activity in the decade following Grover's original proposal for a quantum speed-up of unstructured search, a large discrepancy still remains between the current experimental state of the art and what is required for a practically useful quantum search. To highlight this incongruence, we note that the only physical system in which a scalable quantum search has been performed is in a chain of trapped ions, and in that experiment, the search space totaled just four elements. An

TABLE I. Numerically computed scaled detunings and peak Rabi frequencies for the oracle and the inversion-about-average operators, respectively.

Number of ions	Number of elements	Number of steps	Oracle		Inversion	
(N)	($\mathcal{N}$)	(n_G)	δT	$g_0 T$	δT	$g_0 T$
6	20	3	19.470	28.610	10.320	25.830
8	70	6	21.400	10.800	21.050	24.400
10	252	12	15.687	70.322	88.565	87.142

important intermediate goal on the long road to performing a practically useful quantum search is to demonstrate Grover's algorithm in a moderately sized trapped-ion quantum register. The standard approach of building up the many-ion Grover operator using a network of single- and two-qubit gates is inappropriate for this task, since the required physical resources far exceed those available in today's experiments. By contrast, in this paper, we have proposed to construct the many-ion Grover operator using only two single off-resonant laser pulses, with suitably chosen peak Rabi frequencies and detunings, which synthesize the inversion-about-average and oracle operators, each in a single shot.

A compact recipe for synthesizing the inversion-about-average operator was derived by factorizing the overall Hilbert space into a series of independent ladders. The coupling strengths between the MS states were determined solely through a consideration of the angular momentum structure of the combined ionic pseudospin. The technique proposed in this paper raises the prospect of demonstrating Grover's algorithm in a moderately sized trapped-ion database comprising up to several hundred elements, and which scales exponentially with the number of ions; this is a necessary step on the path to demonstrating a practically useful quantum search, which remains a long-term goal.

ACKNOWLEDGMENTS

This work is supported by the European Commission's projects EMALI and FASTQUAST, the Bulgarian Science Fund Grants No. VU-F-205/06, No. DMU02-19/09, and No. D002-90/08, and the Elite programme of the Landesstiftung-Baden-Württemberg.

[1] L. K. Grover, Phys. Rev. Lett. **79**, 325 (1997).
[2] N. J. Cerf, L. K. Grover, and C. P. Williams, Phys. Rev. A **61**, 032303 (2000).
[3] R. Blume-Kohout, C. M. Caves, and I. H. Deutsch, Found. Phys. **32**, 1641 (2002).
[4] I. L. Chuang, N. Gershenfeld, and M. Kubinec, Phys. Rev. Lett. **80**, 3408 (1998).
[5] J. A. Jones, M. Mosca, and R. H. Hansen, Nature **393**, 344 (1998).
[6] M. S. Anwar, D. Blazina, H. Carteret, S. B. Duckett, and J. A. Jones, Chem. Phys. Lett. **400**, 94 (2004).
[7] P. G. Kwiat, J. R. Mitchell, P. D. D. Schwindt, and A. G. White, J. Mod. Opt. **47**, 257 (2000).
[8] P. Walther, K. J. Resch, T. Rudolph, E. Schenck, H. Weinfurter, V. Vedral, M. Aspelmeyer, and A. Zeilinger, Nature **434**, 169 (2005).
[9] K.-A. Brickman, P. C. Haljan, P. J. Lee, M. Acton, L. Deslauriers, and C. Monroe, Phys. Rev. A **72**, 050306(R) (2005).
[10] J. Ahn, T. C. Weinacht, and P. H. Bucksbaum, Science **287**, 463 (2000).
[11] N. Bhattacharya, H. B. van Linden van den Heuvell, and R. J. C. Spreeuw, Phys. Rev. Lett. **88**, 137901 (2002).
[12] D. Kielpinski, C. Monroe, and D. J. Wineland, Nature **417**, 709 (2002).
[13] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter, Phys. Rev. A **52**, 3457 (1995).
[14] F. Schmidt-Kaler, H. Häfner, M. Riebe, S. Gulde, G. P. T. Lancaster, T. Deuschle, C. Becher, C. F. Roos, J. Eschner, and R. Blatt, Nature **422**, 408 (2003).
[15] D. Leibfried, B. DeMarco, V. Meyer, D. Lucas, M. Barrett, J. Britton, W. M. Itano, B. Jelenkovic, C. Langer, T. Rosenband, and D. J. Wineland, Nature **422**, 412 (2003).
[16] T. Monz, K. Kim, W. Hansel, M. Riebe, A. S. Villar, P. Schindler, M. Chwalla, M. Hennrich, and R. Blatt, Phys. Rev. Lett. **102**, 040501 (2009).
[17] S. S. Ivanov, P. A. Ivanov, and N. V. Vitanov, Phys. Rev. A **78**, 030301(R) (2008).
[18] E. S. Kyoseva, D. G. Angelakis, and L. C. Kwek, Europhys. Lett. **89**, 20005 (2010).
[19] I. E. Linington, P. A. Ivanov, and N. V. Vitanov, Phys. Rev. A **79**, 012322 (2009).
[20] A. S. Householder, J. ACM **5**, 339 (1958).
[21] P. A. Ivanov, E. S. Kyoseva, and N. V. Vitanov, Phys. Rev. A **74**, 022323 (2006).
[22] P. A. Ivanov, B. T. Torosov, and N. V. Vitanov, Phys. Rev. A **75**, 012323 (2007).
[23] P. A. Ivanov and N. V. Vitanov, Phys. Rev. A **77**, 012335 (2008).
[24] E. S. Kyoseva, N. V. Vitanov, and B. W. Shore, J. Mod. Opt. **54**, 2237 (2007).
[25] A. A. Rangelov, N. V. Vitanov, and B. W. Shore, Phys. Rev. A **77**, 033404 (2008).
[26] G. L. Long, Phys. Rev. A **64**, 022307 (2001).
[27] G. L. Long, Y. S. Li, W. L. Zhang, and L. Niu, Phys. Lett. A **262**, 27 (1999).
[28] P. Hoyer, Phys. Rev. A **62**, 052304 (2000).
[29] D. J. Wineland, C. R. Monroe, W. M. Itano, D. Leibfried, B. E. King, and D. M. Meekhof, J. Res. Nat. Inst. Stand. Tech. **103**, 259 (1998).
[30] A. A. Rangelov, N. V. Vitanov, and B. W. Shore, Phys. Rev. A **74**, 053402 (2006).
[31] I. E. Linington and N. V. Vitanov, Phys. Rev. A **77**, 010302(R) (2008).
[32] D. B. Hume, C. W. Chou, T. Rosenband, and D. J. Wineland, Phys. Rev. A **80**, 052302 (2009).
[33] I. E. Linington and N. V. Vitanov, Phys. Rev. A **77**, 062327 (2008).
[34] M. Riebe, H. Häffner, C. F. Roos, W. Hänsel, J. Benhelm, G. P. T. Lancaster, T. W. Körber, C. Becher1, F. Schmidt-Kaler, D. F. V. James, and R. Blatt, Nature **429**, 734 (2004).
[35] N. V. Vitanov, B. W. Shore, L. Yatsenko, K. Böhmer, T. Halfmann, T. Rickes, and K. Bergmann, Opt. Commun. **199**, 117 (2001).
[36] H. Häffner, C. F. Roos, and R. Blatt, Phys. Rep. **469**, 155 (2008).
[37] F. Schmidt-Kaler, S. Gulbe, M. Riebe, T. Deuschle, A. Kreuter, G. Lancaster, C. Becher, J. Eschner, H. Häffner, and R. Blatt, J. Phys. B **36**, 623 (2003).
[38] D. Kielpinski, V. Meyer, M. A. Rowe, C. A. Sackett, W. M. Itano, C. Monroe, and D. J. Wineland, Science **291**, 1013 (2001).