Annuaire de l'Universite de Sofia "St. Kliment Ohridski", Faculte de Physique
Edition Spéciale, 2012

PATH INTEGRAL REPRESENTATION FOR SPIN SYSTEMS

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Наум Карчев. КОНТИНУАЛНИ ИНТЕГРАЛИ ЗА СПИНОВИ СИСТЕМИ

Настоящата статия е кратък обзор на различни представяния чрез континуални интеграли на статистическия интеграл на квантови спинови системи. Първо разглеждам кохерентни състояния за SU(2) алгебра. Различни параметризации на кохерентните състояния водят до различни представяния на континуални интеграли. Всички те са обединени в U(1) калибровъчна теория на квантови спинови системи.

Naoum Karchev. PATH INTEGRAL REPRESENTATION FOR SPIN SYSTEMS

The present paper is a short review of different path integral representations of the partition function of quantum spin systems. To begin with, I consider coherent states for SU(2) algebra. Different parameterizations of the coherent states lead to different path integral representations. They all are unified within an U(1) gauge theory of quantum spin systems.

Keywords: Heiseberg model, Path integral, Coherent states
PACS numbers: 75.10.Jm, 31.15.xk, 75.30.Ds

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1. INTRODUCTION

Path integrals over fields is widely used approach to quantum field theory. I shall discuss the quantum field theory of spin systems.

Spin systems obtain their magnetic properties from a system of localized magnetic moments. The dynamical degrees of freedom are spin-s operators $\mathbf{S}_r$ of localized spins which satisfy the $SU(2)$ algebra. It was shown by Haldane [1, 2] that quantum spin systems could be formulated in terms of path integrals over vectors which identify the local orientation of the spin of the localized electrons.

In the present paper I discuss different path integral representations of the partition function of quantum spin systems.

2. COHERENT STATES FOR $SU(2)$ ALGEBRA

Let S_r^α are the spin operators with $\alpha = 1, 2, 3$ and r labels the lattice sites. They obey the $SU(2)$ algebra

$$\left[S_r^\alpha \, , \, S_{r'}^\beta\right] \; = \; i\delta_{r,r'}\epsilon^{\alpha\beta\gamma}S_r^\gamma \tag{1}$$

We consider the Haisenberg model with a Hamiltonian

$$H \; = \sum_{r,r'} J_{r,r'}\mathbf{S}_r \cdot \mathbf{S}_{r'} \tag{2}$$

where the only condition is $J_{r,r} = 0$.

Let us put in correspondence to any lattice site a $(2s + 1)$ dimensional space of $SU(2)$ group representation. Then the Hilbert space of the system is the tensor product of all these spaces.

Let us choose as a "ground state" $|0\rangle$ the vector which satisfies

$$\S_r^3|0\rangle \; = \; s|0\rangle \tag{3}$$

Following Radcliffe[3] one defines the coherent states and the conjugated coherent states by the relations

$$|z\rangle \; = \; e^{\sum\limits_r z_r \S_r^-}|0\rangle \; = \prod_r e^{z_r S_r^-}|0\rangle \tag{4}$$

$$\langle \bar{z}| = \langle 0|e^{\sum\limits_r S_r^+ \bar{z}_r} = \langle 0|\prod_r e^{S_r^+ \bar{z}_r} \tag{5}$$

where $S_r^{\pm} = S_r^1 \pm i S_r^2$, z_r are complex numbers and $\bar{z}_r$ are their complex conjugated.

The follow formulae for the matrix elements are straightforeward generalization of those given in the Radcliffe's paper [3]

$$\langle \bar{z}'|z\rangle = \prod_r \left(1 + \bar{z}_r' z_r\right)^{2s} \tag{6}$$

$$\langle \bar{z}'|S_r^-|z\rangle = \frac{2s\bar{z}_r'}{(1+\bar{z}_r' z_r)}\langle \bar{z}'|z\rangle \tag{7}$$

$$\langle \bar{z}'|S_r^+|z\rangle = \frac{2s z_r}{(1+\bar{z}_r' z_r)}\langle \bar{z}'|z\rangle \tag{8}$$

$$\langle \bar{z}'|S_r^3|z\rangle = \frac{s\left(1-\bar{z}_r' z_r\right)}{(1+\bar{z}_r' z_r)}\langle \bar{z}'|z\rangle \tag{9}$$

The "resolution of unity", which is an expression of the identity operator in terms of the coherent state operators $|z\rangle\langle\bar{z}|$ is given by

$$\int \prod_r d\mu(z_r) \frac{1}{\prod\limits_r \left(1+\bar{z}_r z_r\right)^{2s}} |z\rangle\langle\bar{z}| = 1 \tag{10}$$

where

$$d\mu(z_r) = \frac{(2s+1)}{\left(1+\bar{z}_r z_r\right)^2}\frac{d^2 z_r}{\pi} \tag{11}$$

and the product $\prod\limits_r$ is over the all lattice sites.

Setting in equations (7),(8) and (9) $z' = z$ one obtains the diagonal matrix elements of the generators $\mathbf{S_r}$

$$\langle \bar{z}|\mathbf{S}_r|z\rangle = s\mathbf{n}_r \langle \bar{z}|z\rangle \tag{12}$$

where $\mathbf{n_r}$ are unit vectors ($\mathbf{n}_r^2 = 1$) given by

$$n_r^1 = \frac{z_r + \bar{z}_r}{1+\bar{z}_r z_r}, \quad n_r^2 = \frac{1}{i}\frac{z_r - \bar{z}_r}{1+\bar{z}_r z_r}, \quad n_r^3 = \frac{1-\bar{z}_r z_r}{1+\bar{z}_r z_r}, \tag{13}$$

The equations (13) map the complex plane onto unit sphere S^2. It is convenient to use the azimuthal angle $0 \leq \theta_r \leq \pi$ and the polar angle $0 \leq \varphi_r \leq 2\pi$ as variables determining the coherent states. Making use of the stereographic projection

$$z_r = tg\frac{\theta_r}{2}\, e^{i\varphi_r} \tag{14}$$

one obtains

$$\mathbf{n}_r = (\cos\varphi_r \sin\theta_r,\ \sin\varphi_r \sin\theta_r,\ \cos\theta_r)$$

Now the equations (7-10) can be rewritten in terms of the two angles. For example the matrix elements (6) take the form

$$\langle \mathbf{n}' | \mathbf{n} \rangle \ = \prod_r e^{i\gamma(\mathbf{n}'_r, \mathbf{n}_r)} \left(\frac{1 + \mathbf{n}'_r \cdot \mathbf{n}_r}{2} \right)^s \tag{15}$$

where $\gamma(\mathbf{n}'_r, \mathbf{n}_r)$ is the area of the spherical triangle with vertices $\mathbf{n}^0 = (0,0,1)$, $\mathbf{n}'_r$ and $\mathbf{n}_r$. The measure (11) is manifestly rotationally invariant if we rewrite it in terms of unit vectors

$$d\mu(\mathbf{n}_r) \ = \ \frac{2s+1}{4\pi} \sin\theta_r d\theta_r d\varphi_r \ = \ \frac{2s+1}{4\pi} \delta(\mathbf{n}_r^2 - 1) d^3 n_r \tag{16}$$

where δ is Dirac's delta function.

Finally, mapping the complex plane onto disk with radius $\sqrt{2s}$ one introduces another parametrization of the coherent states

$$a_r = \frac{\sqrt{2s}\, z_r}{(1 + \bar{z}_r z_r)^{\frac{1}{2}}} \qquad \bar{a}_r = \frac{\sqrt{2s}\, \bar{z}_r}{(1 + \bar{z}_r z_r)^{\frac{1}{2}}} \tag{17}$$

where a_r and $\bar{a}_r$ are complex numbers subject to the condition $\bar{a}_r a_r \leq 2s$. Then the spin vectors $\mathbf{S}_r = s\mathbf{n}_r$ ($\mathbf{S}_r^2 = s^2$) are given by

$$\begin{aligned} S_r^- &= \bar{a}_r \sqrt{2s - \bar{a}_r a_r} \\ S_r^+ &= \sqrt{2s - \bar{a}_r a_r}\ a_r \\ S_r^3 &= s - \bar{a}_r a_r \end{aligned} \tag{18}$$

where $S_r^{\pm} = S_r^1 \pm iS_r^2$, and the measure (11) takes the form

$$d\mu(\bar{a}_r a_r) = \frac{2s+1}{2s}\frac{d\bar{a}_r da_r}{2\pi i} \tag{19}$$

3. PATH INTEGRAL APPROACH FOR HEISENBERG MODEL

Following the path integral approach I use the coherent states in the evaluation of the partition function

$$\mathcal{Z}(\beta) = Tre^{-\beta H}. \tag{20}$$

In equation (20) β is the inverse temperature. It is evident from equation (10) that this function admits the representation

$$\mathcal{Z}(\beta) = \int \prod_r d\mu(z_r) \frac{1}{\prod\limits_r (1+\bar{z}_r z_r)^{2s}} \langle \bar{z}| e^{-\beta H} |z\rangle \tag{21}$$

One may consider the operator $e^{-\beta H}$ as a multiple of many small evolutions

$$e^{-\beta H} = \lim_{N\to\infty} \left(1 - \frac{\beta}{N} H\right)^N \tag{22}$$

Then, using the equation (10) one obtains

$$\begin{aligned} Tre^{-\beta H} \quad = \quad & \lim_{N\to\infty} \int \prod_r d\mu(z_r) \prod_{k=1}^{N-1} d\mu(z_r(\tau_k)) \langle \bar{z}| \left(1 - \frac{\beta}{N} H\right) |z(\tau_{N-1})\rangle \\ & \langle \bar{z}(\tau_{N-1})| \left(1 - \frac{\beta}{N} H\right) |z(\tau_{N-2})\rangle \ldots \langle \bar{z}(\tau_1)| \left(1 - \frac{\beta}{N} H\right) |z\rangle \\ & exp\left\{-2s \sum_r [\ln(1+\bar{z}_r z_r) + \ln(1+\bar{z}_r(\tau_{N-1}) z_r(\tau_{N-1})) \right. \\ & \left. + \ldots + \ln(1+\bar{z}_r(\tau_1) z_r(\tau_1))]\right\} \end{aligned} \tag{23}$$

The kernel $\langle \bar{z}(\tau_k)| \left(1 - \frac{\beta}{N} H\right) |z(\tau_l)\rangle$ can be represented in the form

$$\langle \bar{z}(\tau_k)| \left(1 - \frac{\beta}{N} H\right) |z(\tau_l)\rangle \; = \left(1 - \frac{\beta}{N} h\left(\bar{z}(\tau_k), z(\tau_l)\right)\right) \langle \bar{z}(\tau_k)|z(\tau_l)\rangle$$
$$\simeq \quad exp\left\{-\frac{\beta}{N} h\left(\bar{z}(\tau_k), z(\tau_l)\right) + 2s \sum_r \ln\left(1 + \bar{z}_r z_r\right)\right\}. \tag{24}$$

Making use of the equations (6-9), one represents the Hamiltonian in the form

$$h\left(\bar{z}(\tau_k), z(\tau_l)\right) \; = \; s^2 \sum_{r,r'} J_{r,r'} \; \times \tag{25}$$
$$\frac{2\left[\bar{z}_r(\tau_k) z_{r'}(\tau_l) + \bar{z}_{r'}(\tau_k) z_r(\tau_l)\right] + \left[1 - \bar{z}_r(\tau_k) z_r(\tau_l)\right]\left[1 - \bar{z}_{r'}(\tau_k) z_{r'}(\tau_l)\right]}{\left[1 + \bar{z}_r(\tau_k) z_r(\tau_l)\right]\left[1 + \bar{z}_{r'}(\tau_k) z_{r'}(\tau_l)\right]}$$

where the term independent of $\bar{z}_r(\tau_k)$ and $z_r(\tau_l)$ is dropped.

Now we proceed taking the continuum limit $N \to \infty$ and find the path integral representation of the partition function

$$\mathcal{Z}(\beta) \; = \; \int \prod_{\tau, r} d\mu(z_r(\tau)) e^{-S(\bar{z}, z)} \tag{26}$$

where

$$S(\bar{z}, z) \; = \; \int_0^\beta d\tau \left\{2s \sum_r \frac{1}{1 + \bar{z}_r(\tau) z_r(\tau)} \bar{z}_r \dot{z}_r + h(\bar{z}(\tau), z(\tau))\right\} \tag{27}$$

is the action, and the hamiltonian is

$$h(\bar{z}(\tau), z(\tau)) \; = \; s^2 \sum_{r,r'} J_{r,r'} \mathbf{n}_r(\tau) \cdot \mathbf{n}_{r'}(\tau) \tag{28}$$

In the above, the overdots correspond to time derivatives. The complex fields $\bar{z}_r(\tau), z_r(\tau)$, and the real vector fields $\mathbf{n}_r(\tau)$ satisfy periodic boundary conditions $\bar{z}_r(\beta) = \bar{z}_r(0), z_r(\beta) = z_r(0), \mathbf{n}_r(0) = \mathbf{n}_r(\beta)$.

One can use the coherent states labeled by the unit vector $\mathbf{n}_r$ to derive the path integral representation for the partition function [1, 2]. Then the measure is given by

$$d\mu(\mathbf{n}) = \prod_{\tau,r} \frac{2s+1}{4\pi} d^3 n_r(\tau) \delta(\mathbf{n}_r^2(\tau) - 1) \tag{29}$$

(see Eq.(16)) and the action adopts the form

$$S = \int_0^\beta d\tau \left[is \sum_r \mathbf{A}(\mathbf{n}_r) \cdot \dot{\mathbf{n}}_r(\tau) + h(\tau) \right] \tag{30}$$

In equation (30) $\mathbf{A}(\mathbf{n}_r)$ is the vector potential of a Dirac magnetic monopole at the center of the unit sphere

$$\mathbf{A} = \frac{1-\cos\theta}{\sin\theta} \mathbf{e}_\varphi \tag{31}$$

It obeys locally

$$\partial_{\mathbf{n}} \times \mathbf{A}(\mathbf{n}) = \mathbf{n} \tag{32}$$

The kinetic term in Eq.(30) is invariant under the gauge transformations

$$\mathbf{A} \to \mathbf{A} + \partial_{\mathbf{n}} \alpha,$$

where the parameter α is defined on the sphere. It is more convenient for further calculations to do a gauge transformation which leads to the vector potential

$$\mathbf{A} = -\coth\theta \ \mathbf{e}_\varphi \tag{33}$$

In this case the half of the string is up the north pole and the other half is down the south pole. Thus, the vector potential is an even function of its argument

$$\mathbf{A}(\mathbf{n}) = \mathbf{A}(-\mathbf{n}) \tag{34}$$

Making use of the third parametrization of the coherent states (17) one obtains a path integral in terms of complex fields $a_r(\tau)$ and $\bar{a}_r(\tau)$, which satisfy $\bar{a}_r(\tau)a_r(\tau) \leq 2s$ and periodic boundary conditions. The measure is given by

$$d\mu(\bar{a}, a) = \prod_{\tau,r} \frac{2s+1}{2s} \frac{d\bar{a}_r(\tau) da_r(\tau)}{2\pi i} \tag{35}$$

Substituting in the Hamiltonian and the kinetic term one rewrites the action in the form

$$S = \int_0^\beta d\tau \left[\frac{1}{2} \sum_r \left(\bar{a}_r(\tau)\dot{a}_r(\tau) - \dot{\bar{a}}_r(\tau)a_r(\tau) \right) + \sum_{r,r'} J_{r,r'} \mathbf{S}_r(\tau) \cdot \mathbf{S}_{r'}(\tau) \right] \tag{36}$$

where the spin vectors are given by (2.18).

The field theories defined by the actions (27), (30) and (36) can be thought of as a particular case of a more general Abelian gauge field theory. To see how does this come about I introduce two complex fields $\psi_r^k(\tau)$ $(k = 1, 2)$ by the relations

$$\begin{aligned} |\psi_r^1(\tau)| &= \frac{\sqrt{2s}}{(1 + \bar{z}_r(\tau)z_r(\tau))^{\frac{1}{2}}} = (2s - \bar{a}_r(\tau)a_r(\tau))^{\frac{1}{2}} \\ \psi_r^2(\tau) &= \frac{\sqrt{2s}\, z_r(\tau)}{(1 + \bar{z}_r(\tau)z_r(\tau))^{\frac{1}{2}}} = a_r(\tau) \end{aligned} \tag{37}$$

Let substitute Eq.(37) in Eq.(27) or Eq.(36) and take into account the condition

$$\arg \psi_r^1(\tau) = 0 \tag{38}$$

One gets

$$S = \int_0^\beta d\tau \left[\sum_{\alpha,r} \bar{\psi}_r^\alpha(\tau) \frac{d}{d\tau} \psi_r^\alpha(\tau) + \sum_{r,r'} J_{r,r'} \mathbf{S}_r(\tau) \cdot \mathbf{S}_{r'}(\tau) \right] \tag{39}$$

where the spin vectors are equal to

$$S_r^\nu(\tau) = \frac{1}{2} \bar{\psi}_r^\alpha(\tau)\, \sigma_{\alpha\alpha'}^\nu \psi_r^{\alpha'}(\tau) \tag{40}$$

and σ^ν are Pauli matrices. The new fields obey the constraint [4]

$$\bar{\psi}_r^1(\tau)\psi_r^1(\tau) + \bar{\psi}_r^2(\tau)\psi_r^2(\tau) = 2s. \tag{41}$$

Let take care of the local constraint (41) by introducing an extra term in the action with the Lagrangian multiplier field $\lambda_r(\tau)$ which enforces the constraint. Collecting all terms one obtains the final expression for the partition function

$$\mathcal{Z}(\beta) = \int \prod_{\alpha,r,\tau} d\bar{\psi}_r^\alpha(\tau) d\psi_r^\alpha(\tau) \prod_{r,\tau} \delta\left(arg\psi_r^1(\tau)\right) e^{-S_{tot}} \tag{42}$$

where the action

$$S_{tot} = \int_0^\beta d\tau \left[\sum_{\alpha,r} \bar{\psi}_r^\alpha(\tau)\left(\frac{d}{d\tau} - i\lambda_r(\tau)\right)\psi_r^\alpha(\tau) + 2si\lambda_r(\tau) \right.$$
$$\left. + \sum_{r,r'} J_{r,r'} \mathbf{S}_r(\tau) \cdot \mathbf{S}_{r'}(\tau)\right] \tag{43}$$

is invariant under the gauge transformations

$$\begin{aligned} \psi_r^{\alpha\prime}(\tau) &= e^{i\gamma_r(\tau)}\psi_r^\alpha(\tau), \quad \bar{\psi}_r^{\alpha\prime}(\tau) = e^{-i\gamma_r(\tau)}\bar{\psi}_r^\alpha(\tau), \\ \lambda_r'(\tau) &= \lambda_r(\tau) + \frac{d}{d\tau}\gamma_r(\tau) \end{aligned} \tag{44}$$

if the gauge parameters $\gamma_r(\tau)$ satisfy $\gamma_r(0) = \gamma_r(\beta)$.

Let us discuss the Abelian gauge theory (43). Following the standard procedure of quantization, one has to impose an additional gauge fixing condition. If we do this, imposing the condition (38), and then solve the constraint (41), using different parameters we get different field theoretical realizations of the Heisenberg model. On the other hand, one can choose the gauge fixing condition in an alternative way. For example, a convenient gauge condition is the temporal condition imposed on the Lagrangian multiplier $\lambda_r(\tau)$. The gauge fixing condition reads

$$\lambda_r(\tau) = \lambda_r \tag{45}$$

where λ_r depends on the lattice sites but is not a function of the imaginary time. It follows from derivation, that so obtained field-theoretical descriptions are equivalent.

4. SUMMARY

Starting from different parameterizations of coherent states of $SU(2)$ algebra I have derived different path integral representations for the quantum spin systems. In the first case Eqs.(26,27) the path integral is over complex fields and the representation is appropriate for Monte Carlo numerical calculations. In the second case Eqs. (29,30), the path integral is over unite vectors which identify the local orientation of the spin of the localized electrons. For antiferromagnetic systems it is utilized to derive [1] an effective σ model of the antiferromagnetism. For these calculations one uses the representation Eq.(33) of the Dirac's vector field. Finally, the path integral over the complex fields $\bar{a}_r(\tau)$ and $a_r(\tau)$ Eqs.(35,36) is an alternative to the operator Holstein-Primakoff approach.

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